JEE

Class Companion
Mathematics

For JEE Main and Advanced

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CHAPTER

4

Trigonometric Equation (Phase–II)

Learning Objectives

After studying this chapter, you will be able to:

4-1. learn solution of trigonometric equation

4-2. learn solution of quadratic equation

Solution of Trigonometric Equations

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g., if $\sin\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as:

1. Principal solution
2. General solution.

Principal Solutions

The solutions of trigonometric equation which lie in the interval $[0, 2\pi)$ are called *principal solutions*.

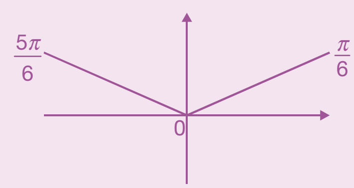
EXAMPLE 1

Find the principal solutions of the equation $\sin x = \frac{1}{2}$.

SOLUTION

$\because \quad \sin x = \frac{1}{2} \quad \because$ there exists two values

i.e., $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$



$\therefore$ Principal solutions of the equation $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$

General Solution

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called **General Solution**.

1. If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.
2. If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.
3. If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.
4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
5. If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
6. If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

Note

α is called the principal angle.

EXAMPLE 2

Solve: $\sec 2\theta = -\frac{2}{\sqrt{3}}$.

SOLUTION

$\sec 2\theta = -\frac{2}{\sqrt{3}} \quad \Rightarrow \quad \cos 2\theta = -\frac{\sqrt{3}}{2}$

$\Rightarrow \cos 2\theta = \cos\frac{5\pi}{6} \quad \Rightarrow \quad 2\theta = 2n\pi \pm \frac{5\pi}{6}, n \in I$

$\Rightarrow \theta = n\pi \pm \frac{5\pi}{12}, n \in I.$

EXAMPLE 3

Solve: $\tan\theta = 2$.

SOLUTION

$\tan\theta = 2$ **(1)**

Let $2 \tan\alpha \quad \Rightarrow \quad \tan\theta = \tan\alpha$

$\Rightarrow \theta = n\pi + \alpha$, where $\alpha = \tan^{-1}(2)$, $n \in I$.

EXAMPLE 4

Solve: $\cos^2\theta = \frac{1}{2}$.

SOLUTION

$\cos^2\theta = \frac{1}{2} \quad \Rightarrow \quad \cos^2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 \quad \Rightarrow \quad \cos^2\theta = \cos^2\frac{\pi}{4}$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}, n \in I$

EXAMPLE 5

Solve: $4\tan^2\theta = 3\sec^2\theta$.

SOLUTION

$\because \quad 4\tan^2\theta = 3\sec^2\theta$ **(1)**

For equation (1) to be defined $\theta \neq (2n + 1)$, $n \in I$.

$\because$ equation (1) can be written as: $\frac{4\sin^2\theta}{\cos^2\theta} = \frac{3}{\cos^2\theta}$

$\because \quad \theta \neq (2n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow \quad 4\sin^2\theta = 3 \quad \therefore \quad \cos^2\theta \neq 0$

$\Rightarrow \quad \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 \quad \Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3}$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3}, n \in I.$

Solutions of Equations by Factorising

EXAMPLE 6

Solve: $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$.

SOLUTION

$\frac{5}{2}$ $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow \quad (2\sin x - \cos x)(1 + \cos x) - (1 - \cos x)(1 + \cos x) = 0$

$\Rightarrow \quad (1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow \quad 1 + \cos x = 0 \quad$ or $\quad 2\sin x - 1 = 0$

$\Rightarrow \quad \cos x = -1 \quad$ or $\quad \sin x = \frac{1}{2}$

$\Rightarrow \quad x = (2n + 1)\pi, n \in I \quad \text{or} \quad \sin x = \sin(\frac{\pi}{6}), n \in I$

$\Rightarrow \quad x = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is

$(2n + 1)\,\pi, n \in I \quad \text{or} \quad n\pi + (-1)^n \frac{\pi}{6}, n \in I.$

EXAMPLE 7

Solve the equation: $\sin^3 x \cos x - \sin x \cos^3 x = \frac{1}{4}$.

SOLUTION

The equation can be written as $4 \sin x \cos x (\sin^2 x - \cos^2 x) = 1$,

$\Rightarrow \quad -2 \sin 2x \cos 2x = -\sin 4x = 1$

$\Rightarrow \quad x = -\frac{3\pi}{2} + k\cdot\frac{\pi}{2} \; (k = 0, \pm 1, \pm 2, ...).$

EXAMPLE 8

Find the general solution of the equation

$$\frac{1+\sin x+\sin^2 x+\ldots+\sin^n x+\ldots\infty}{1-\sin x+\sin^2 x-\sin^3 x+\ldots(-1)^n \sin^n x+\ldots\infty} = \frac{4}{1+\tan^2 x}$$

where $x \neq k\pi + \frac{\pi}{2}, k \in I.$

SOLUTION

N^r of LHS $= \frac{1}{1-\sin x}$; $\quad D^r$ of LHS $= \frac{1}{1+\sin x}$

hence, $\frac{1+\sin x}{1-\sin x} = \frac{4}{\sec^2 x} = 4\cos^2 x$

$= 4(1 - \sin x)(1 + \sin x)$

hence, $4(1 - \sin x)^2 = 1 \quad \Rightarrow \quad (1 - \sin x)^2 = \frac{1}{4}$

$\Rightarrow \quad (1 - \sin x) = \frac{1}{2} \text{ or } -\frac{1}{2}$

$\therefore \quad \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = \frac{3}{2}$ (rejected)

$\therefore \quad \sin x = \sin\frac{\pi}{6} \quad \Rightarrow \quad x = n\pi + (-1)^n \frac{\pi}{6}, n \in I.$

EXAMPLE 9

Find the general solution of the equation
$\sin^3 x(1 + \cot x) + \cos^3 x(1 + \tan x) = \cos 2x.$

SOLUTION

$\sin^2 x(\cos x + \sin x) + \cos^2 x (\cos x + \sin x) = \cos 2x$
$(\cos x + \sin x)(\cos^2 x + \sin^2 x) = (\cos x + \sin x)(\cos x - \sin x)$

$\therefore \quad (\cos x + \sin x)[\cos x - \sin x - 1)] = 0$

$\therefore$ either $\quad \cos x + \sin x = 0 \quad$ **(1)**

or $\quad \cos x - \sin x = 1 \quad$ **(2)**

from (1) $\tan x = -1$ or $1 - \sin 2x = 1 \Rightarrow \sin 2x = 0$

If $\tan x = -1 = \tan\left(-\frac{\pi}{4}\right) \quad \therefore \quad x = n\pi - \frac{\pi}{4}, n \in I.$

If $\sin 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \frac{n\pi}{2}$ this is to be rejected because of the tan x or cot x will not be defined so $x = \left(n\pi - \frac{\pi}{4}\right), n \in I.$

EXAMPLE 10

Find the solutions of the equation, $\log_{\sqrt{2}\sin x}(1+\cos x) = 2$ in the interval $x \in [0, 2\pi]$.

SOLUTION

$2\sin^2 x = 1 + \cos x;\; 2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \quad \cos x = \frac{1}{2} \quad \text{or} \quad -1 \quad \Rightarrow \quad x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

but $\quad x = \pi$ and $\frac{5\pi}{3}$ are rejected $\quad \Rightarrow \quad x = \frac{\pi}{3}.$

Solutions of Equations Reducible to Quadratic Equations

EXAMPLE 11

Solve the equation

$$\sin^2 x(\tan x + 1) = 3\sin x(\cos x - \sin x) + 3.$$

SOLUTION

The given equation makes no sense when $\cos x = 0$; therefore, we can suppose that $\cos x \neq 0$. Noting that the right-hand member of the equation is equal to 3

$\sin x \cos x + 3\cos^2 x$, and dividing both members by $\cos^2 x$, we obtain $\tan^2 x(\tan x + 1) = 3(\tan x + 1)$,

$\Rightarrow \quad (\tan^2 x - 3)(\tan x + 1) = 0$

$\Rightarrow \quad x_1 = -\dfrac{\pi}{4} + k\pi,\ x_2 = \dfrac{\pi}{3} + k\pi,\ x_3 = -\dfrac{\pi}{3} + k\pi.$

EXAMPLE 12

Find the general solution set of the equation $\log_{\tan x}(2 + 4\cos^2 x) = 2$.

SOLUTION

$2 + 4\cos^2 x = \tan^2 x$

$\Rightarrow \quad 2 + 4\cos^2 x = \tan^2 x$

$\Rightarrow \quad 4\cos^4 x + 3\cos^2 x - 1 = 0$

let $\cos^2 x = t \quad \Rightarrow \quad 4t^2 + 3t - 1 = 0$

$\Rightarrow \quad (4t - 1)(t + 1) = 0 \quad \Rightarrow \quad t = \dfrac{\pi}{7}$ or $t = -1$

$\Rightarrow \quad \cos^2 x = \dfrac{\pi}{7}$ or $\cos^2 x = -1$ (not possible)

$\Rightarrow \quad \cos^2 x = \cos^2 \dfrac{\pi}{3} \quad \Rightarrow \quad x = n\pi + \dfrac{\pi}{3},\ n \in I.$

EXAMPLE 13

The equation $\cos^2 x - \sin x + a = 0$ has roots when $x \in \left(0, \dfrac{\pi}{2}\right)$ find 'a'.

SOLUTION

$1 - \sin^2 x - \sin x + a = 0$

$\Rightarrow \quad \sin^2 x + \sin x - (a + 1) = 0 \qquad (\text{let } \sin x = t)$

$\therefore \quad t^2 + t - (a + 1) = 0,\ t \in (0, 1)$

$\Rightarrow \quad t = \dfrac{-1 \pm \sqrt{1 + 4(a + 1)}}{2}$

$\Rightarrow \quad t = \dfrac{-1 \pm \sqrt{4a + 5}}{2}$ (reject – ve sign)

$\therefore \quad t = \dfrac{-1 + \sqrt{4a + 5}}{2}$ now $0 < \dfrac{-1 + \sqrt{4a + 5}}{2} < 1$

$\Rightarrow \quad 0 < -1 + \sqrt{4a + 5} < 2$ or $1 < \sqrt{4a + 5} < 3$

$\Rightarrow \quad 1 < 4a + 5 < 9 \quad \Rightarrow \quad -4 < 4a < 4$

$\Rightarrow \quad -1 < a < 1 \quad \Rightarrow \quad a \in (-1, 1).$

EXAMPLE 14

Solve the equation $\sin^8 x + \cos^8 x = \dfrac{17}{32}$.

SOLUTION

Using the identity $(\sin^2 x + \cos^2 x)^2 = 1$ we get $\sin^4 x + \cos^4 x = 1 - \dfrac{1}{2}\sin^2 2x$,

whence, $\sin^8 x + \cos^8 x = \left(1 - \dfrac{1}{2}\sin^2 2x\right)^2 - \dfrac{1}{8}\sin^4 2x = \dfrac{17}{32}$

$\Rightarrow \quad 1 - \sin^2 2x + \dfrac{1}{8}\sin^4 2x = \dfrac{17}{32}$

$\Rightarrow \quad \sin^4 2x - 8\sin^2 2x + \dfrac{15}{4} = 0.$

Solving we get $\sin^2 2x = 4 \pm \dfrac{7}{2}$, $\sin^2 2x = \dfrac{1}{2}$, $2x = \dfrac{\pi}{4} + k\dfrac{\pi}{2}$; whence, $x = \dfrac{2k + 1}{8}\pi$.

EXAMPLE 15

Find all solutions of the equation $(\tan^2 x - 1)^{-1} = 1 + \cos 2x$, which satisfy the inequality $2^{x+1} - 8 > 0$.

SOLUTION

Let us reduce the initial trigonometric equation to the form $(1 + \cos 2x)\left(1 + \dfrac{1}{2\cos 2x}\right) = 0.$

The following values of x are solutions of this equation $x = -\dfrac{\pi}{2} + \pi n,\ x = \pm\dfrac{\pi}{3} + \pi k,\ n, k \in Z.$

By the hypothesis, we must choose those values of x which satisfy the inequalities

$2^{x+1} - 8 > 0, \cos x \neq 0$. The values we need are $x = \pm\dfrac{\pi}{3} + \pi n$, $n \in N$.

EXAMPLE 16

Determine all the values of a for which the equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ is solvable. Find the solutions.

SOLUTION

Applying the formula $\sin^4 x = \left(\dfrac{1 - \cos 2x}{2}\right)^2$, $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ and putting $\cos 2x = t$

we rewrite the given equation in the form

$$t^2 - 6t + 4a^2 - 3 = 0 \quad (1)$$

The original equation has solutions for a given value of a if and only if, for his value of a, the roots t_1 and t_2 of the equation (1) are real and at least one of these roots does not exceed unity in its absolute value. Solving equation (1), we find $t_1 = 3 - 2\sqrt{3-a^2}$, $t_2 = 3 + 2\sqrt{3-a^2}$.

Hence, the roots of equation (1) are real if

$$|a| \le \sqrt{3} \quad (2)$$

If condition (2) is fulfilled, then $t_2 > 1$ and, therefore, this root can be dicarded. Thus, the problem is reduced to finding the values of a satisfying condition (2), for which $|t_1| \le 1$, i.e.,

$$-1 \le 3 - 2\sqrt{3-a^2} \le 1. \quad (3)$$

From (3) we find $-4 \le -2\sqrt{3-a^2} \le -2$,

$$\Rightarrow \quad 2 \ge \sqrt{3-a^2} \ge 1. \quad (4)$$

Since the inequality $2 \ge \sqrt{3-a^2}$ is fulfilled for $|a| \le \sqrt{3}$, the system of inequalities (4) is reduced to the inequality $\sqrt{3-a^2} \ge 1$, $\Rightarrow$ $|a| \le \sqrt{2}$.

Thus, the original equation is solvable if $|a| \le \sqrt{2}$, and its solutions are

$$x = \pm\frac{1}{2}\text{arc cos }(3 - 2\sqrt{3-a^2}) + k\pi.$$

Solving Equations by Introducing an Auxiliary Argument

EXAMPLE 17

Solve: $\sin x + \cos x = \sqrt{2}$.

SOLUTION

$\because \quad \sin x + \cos x = \sqrt{2} \quad (1)$

Here, $a = 1, b = 1$

$\therefore$ divide both sides of equation (1) by $\sqrt{2}$,

we get $\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = 1$

$\Rightarrow \sin x \cdot \sin\frac{\pi}{4} + \cos x \cdot \cos\frac{\pi}{4} = 1$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow x - \frac{\pi}{4} = 2n\pi, n \in I \quad \Rightarrow \quad x = 2n\pi + \frac{\pi}{4}, n \in I$

$\therefore$ Solution of given equation is $2n\pi + \frac{\pi}{4}, n \in I$.

EXAMPLE 18

Solve the equation $\cos 7x - \sin 5x = \sqrt{3}(\cos 5x - \sin 7x)$.

SOLUTION

Rewrite the equation in the form $\frac{1}{2}\cos 7x + \frac{\sqrt{3}}{2}\sin 7x = \frac{\sqrt{3}}{2}\cos 5x + \frac{1}{2}\sin 5x$

or

$\sin\frac{\pi}{6}\cos 7x + \cos\frac{\pi}{6}\sin 7x = \sin\frac{\pi}{3}\cos 5x + \cos\frac{\pi}{3}\sin 5x$, i.e., $\sin\left(\frac{\pi}{6} + 7x\right) = \sin\left(\frac{\pi}{3} + 5x\right)$.

But $\sin\alpha = \sin\beta$ if and only if either $\alpha - \beta = 2k\pi$ or $\alpha + \beta = (2m+1)\pi$ $(k, m = 0, \pm 1, \pm 2, ...)$.

Hence, $\frac{\pi}{6} + 7x - \frac{\pi}{6} - 5x = 2k\pi$ or $\frac{\pi}{6} + 7x - \frac{\pi}{3} - 5x = (2m+1)\pi$.

Thus, the roots for the equation are

$$\left.\begin{aligned} x &= \frac{\pi}{12}(12k+1) \\ x &= \frac{\pi}{24}(4m+1) \end{aligned}\right\} (k, m = 0, \pm 1, \pm 2, ...).$$

EXAMPLE 19

Solve the equation $2\sin 17x + \sqrt{3}\cos 5x + \sin 5x = 0$.

SOLUTION

Dividing both sides of the equation by 2, we reduce it to the form $\sin 17x + \sin\left(5x + \frac{\pi}{3}\right) = 0$,

whence, we obtain $2\sin\left(11x + \frac{\pi}{6}\right)\cos\left(6x - \frac{\pi}{6}\right) = 0$

$\Rightarrow \quad x_1 = -\frac{\pi}{66} + \frac{k\pi}{11},\ x_2 = \frac{\pi}{36} + \frac{(2k+1)\pi}{12}$.

EXAMPLE 20

Solve the equation $\sin^3 x + \cos^3 x = 1 - \frac{1}{2}\sin 2x$.

SOLUTION

Using the formula for the sum of cubes of two members we transform the Left-hand Side (L.H.S) of the equation in the following way: $(\sin x + \cos x)(1 - \sin x \cos x)$

$= \left(1 - \frac{1}{2}\sin 2x\right)(\sin x + \cos x)$.

Hence, the original equation takes the form

$$\left(1 - \frac{1}{2}\sin 2x\right)(\sin x + \cos x - 1) = 0.$$

The expression in the first brackets is different from zero for all x. Therefore, it is sufficient to consider the equation $\sin x + \cos x - 1 = 0$. The latter is reduced to the form

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad x_1 = 2\pi k,\ x_2 = \frac{\pi}{2} + 2\pi k.$$

Solving Equations by Transforming a Sum of Trigonometric Functions Into a Product

EXAMPLE 21

Solve: $\cos 3x + \sin 2x - \sin 4x = 0$.

SOLUTION

$\cos 3x + \sin 2x - \sin 4x = 0$

$\Rightarrow \quad \cos 3x + 2\cos 3x \cdot \sin(-x) = 0$

$\Rightarrow \quad \cos 3x - 2\cos 3x \cdot \sin x = 0$

$\Rightarrow \quad \cos 3x(1 - 2\sin x) = 0$

$\Rightarrow \quad \cos 3x = 0 \quad \text{or} \quad 1 - 2\sin x = 0$

$\Rightarrow \quad 3x = (2n+1)\frac{\pi}{2},\ n \in I \quad \text{or} \quad \sin x = \frac{1}{2}$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{6},\ n \in I \quad \text{or} \quad x = n\pi + (-1)^n\frac{\pi}{6},\ n \in I$

$\therefore$ Solution of given equation is

$(2n+1)\frac{\pi}{6},\ n \in I \quad \text{or} \quad n\pi + (-1)^n\frac{\pi}{6},\ n \in I.$

Solving Equations by Transforming a Product of Trigonometric Functions Into a Sum

EXAMPLE 22

Solve: $\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$.

SOLUTION

$\because \quad \sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$

$\Rightarrow \quad 2\sin 5x \cdot \cos 3x = 2\sin 6x \cdot \cos 2x$

$\Rightarrow \quad \sin 8x + \sin 2x = \sin 8x + \sin 4x$

$\Rightarrow \quad \sin 4x - \sin 2x = 0$

$\Rightarrow \quad 2\sin 2x \cdot \cos 2x - \sin 2x = 0$

$\Rightarrow \quad \sin 2x(2\cos 2x - 1) = 0$

$\Rightarrow \quad \sin 2x = 0 \quad \text{or} \quad 2\cos 2x - 1 = 0$

$\Rightarrow \quad 2x = n\pi,\ n \in I \quad \text{or} \quad \cos 2x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{n\pi}{2},\ n \in I \quad \text{or} \quad 2x = 2n\pi \pm \frac{\pi}{3},\ n \in I$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is

$\frac{n\pi}{2}, n \in I$ or $n\pi \pm \frac{\pi}{6}, n \in I.$

EXAMPLE 23

Solve the equation $\cot x - 2 \sin 2x = 1$.

SOLUTION

1. The equation becomes senseless for $x = k\pi$. For all the other values of x it is equivalent to the equation $\cos x - \sin x = 2 \sin 2x \cdot \sin x$ we obtain $\cos x - \sin x = \cos x - \cos 3x$, $\sin x = \cos 3x$,

   whence, $\sin x = \sin\left(\frac{\pi}{2} - 3x\right)$. Consequently,

   $2 \sin\left(2x - \frac{\pi}{4}\right)\cos\left(x - \frac{\pi}{4}\right) = 0$

   $\Rightarrow \quad x_1 = \frac{\pi}{8} + \frac{k\pi}{2}, x_2 = \frac{4\pi}{4} + k\pi\cdot$

2. Putting $\tan x = t$, we get the equation $t^3 + 3t^2 + t - 1 = 0$.

   Factoring the left member, we obtain $(t + 1)(t + 1 - \sqrt{2})(t + 1 + \sqrt{2}) = 0.$

   hence, $(\tan x_1) = 1$, $(\tan x_2) = \sqrt{2} - 1$, $(\tan x_3) = -1 - \sqrt{2}$.

   $\Rightarrow \quad x_1 = \frac{3\pi}{4} + k\pi; x_2 = \text{arc tan}(\sqrt{2} - 1) + k\pi,$

   $x_3 = -\text{arc tan}(1 + \sqrt{2}) + k\pi.$

Solving Equations by a Change of Variable

1. Equations of the form $P(\sin x \pm \cos x, \sin x \cdot \cos x) = 0$, where $P(y, z)$ is a polynomial, can be solved by the change $\cos x \pm \sin x = t \Rightarrow 1 \pm 2 \sin x \cdot \cos x = t^2$.
2. Equations of the form of $a \cdot \sin x + b \cdot \cos x + d = 0$, where a, b and d are real numbers and $a, b \neq 0$ can be solved by changing $\sin x$ and $\cos x$ into their corresponding tangent of half the angle.
3. Many equations can be solved by introducing a new variable. e.g. the equation $\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to $2(y + 1)\left(y - \frac{1}{2}\right) = 0$ by substituting, $\sin 2x \cdot \cos 2x = y$.

EXAMPLE 24

Solve $\sin x + \cos x = 1 \sin x \cdot \cos x$.

SOLUTION

$\because \quad \sin x + \cos x = 1 \sin x \cdot \cos x \quad$ **(1)**

Let $\sin x + \cos x = t$

$\Rightarrow \quad \sin^2 x + \cos^2 x + 2 \sin x \cdot \cos x = t^2$

$\Rightarrow \quad \sin x \cdot \cos x = \frac{t^2 - 1}{2}$

Now put $\sin x + \cos x = t$ and $\sin x \cdot \cos x = \frac{t^2 - 1}{2}$ in (1), we get $t = 1 + \frac{t^2 - 1}{2}$

$\Rightarrow \quad t^2 - 2t + 1 = 0 \quad \Rightarrow \quad t = 1 \quad (\because t = \sin x + \cos x)$

$\Rightarrow \quad \sin x + \cos x = 1 \quad$ **(2)**

divide both sides of equation (2) by $\sqrt{2}$, we get

$\Rightarrow \quad \sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

$\Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

(a) if we take positive sign, we get $x = 2n\pi + \frac{\pi}{4}$, $n \in I$.

(b) if we take negative sign, we get $x = 2n\pi$, $n \in I$.

EXAMPLE 25

Solve the equation $\sin 2x - 12(\sin x - \cos x) + 12 = 0$.

SOLUTION

Putting $\sin x - \cos x = t$ and using the identity $(\sin x - \cos x)^2 = 1 - 2\sin x \cos x$, we rewrite the original equation in the form $t^2 + 12t - 13 = 0$.

This equation has the roots $t_1 = -13$ and $t_2 = 1$.

But $t = \sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$, and thus, $|t| \le \sqrt{2}$. Consequently, the root $t_1 = -13$ must be discarded. Therefore, the original equation is reduced to the equation $\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$$\Rightarrow \quad x_1 = \pi + 2k\pi,\ x_2 = \frac{\pi}{2} + 2k\pi.$$

EXAMPLE 26

Solve the equation $1 + 2\operatorname{cosec} x = -\dfrac{\sec^2 \frac{x}{2}}{2}$.

SOLUTION

Transform the given equation to the form $2\cos^2 \frac{x}{2}(2 + \sin x) + \sin x = 0$.

Using the formula $2\cos^2 \frac{x}{2} = 1 + \cos x$ and opening the brackets, we obtain

$$2 + 2(\sin x + \cos x) + \sin x \cdot \cos x = 0 \qquad \textbf{(1)}$$

By the substitution $\sin x + \cos x = t$ equation (1), is reduced to the quadratic equation $t^2 + 4t + 3 = 0$ whose roots are $t_1 = -1$ and $t_2 = -3$. Since $|\sin x + \cos x| \le \sqrt{2}$, the original equation can only be satisfied by the roots of the equation

$$\sin x + \cos x = -1 \qquad \textbf{(2)}$$

Solving equation (2), we obtain $x_1 = -\frac{\pi}{2} + 2k\pi$ and $x_2 = (2k + 1)\pi$.

Here, x_2 should be discarded because $\sin x_2 = 0$, and therefore, the original equation makes no sense for

$$x = x_2 \quad \Rightarrow \quad x = -\frac{\pi}{2} + 2k\pi.$$

EXAMPLE 27

Solve the equation $\sin x + \cos x - 2\sqrt{2}\sin x \cos x = 0$.

SOLUTION

Designating $\sin x + \cos x = t$ and using the equation $\sin x \cos x = \frac{(t^2 - 1)}{2}$, we reduce the equation to a new equation with respect to t: $\sqrt{2}t^2 - t - \sqrt{2} = 0$.

The numbers $t_1 = \sqrt{2}$, $t_2 = -\frac{1}{\sqrt{2}}$ are roots of this quadratic equation.

Thus, the solution of the initial equation reduces to the solution of the trigonometric equations:

$$\sin x + \cos x = \sqrt{2},\ \sin x + \cos x = -\frac{1}{\sqrt{2}}.$$

Multiplying both sides of these equations by the number $\frac{1}{\sqrt{2}}$, we reduce them to two simpler equations:

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$$

$$\Rightarrow \ \sin x \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = 1 \ \Rightarrow \ \sin\left(x + \frac{\pi}{4}\right) = 1.$$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2} \quad \Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{2}.$$

The solutions of the equations $\sin\left(x + \frac{\pi}{4}\right) = 1$ and $\sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{2}$ are

$$x = \frac{\pi}{4} + 2\pi k,\ k \in Z;\ x = (-1)^{n+1}\frac{\pi}{6} - \frac{\pi}{4} + \pi n,\ n \in Z.$$

EXAMPLE 28

Solve the equation

$$\frac{1}{2}(\sin^4 x + \cos^4 x) = \sin^2 x \cos^2 x + \sin x \cos x.$$

SOLUTION

We obtain the equation $\sin^2 2x + \sin 2x - 1 = 0$.

Solving it, we get $\sin 2x = \frac{\sqrt{5} - 1}{2}$

$$\Rightarrow \quad x = (-1)^k \frac{1}{2}\arcsin\frac{\sqrt{5} - 1}{2} + \frac{k\pi}{2}.$$

EXAMPLE 29

Solve: 3 cos x + 4 sin x = 5.

SOLUTION

$\because \quad 3\cos x + 4\sin x = 5$

$\because \quad \cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ **(1)**

$\therefore$ equation (1) becomes

$\Rightarrow \quad 3\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+4\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)=5$ **(2)**

Let $\tan\frac{x}{2}=t$

$\therefore$ equation (1) becomes $3\left(\dfrac{1-t^2}{1+t^2}\right)+4\left(\dfrac{2t}{1+t^2}\right)=5$

$\Rightarrow \quad 4t^2 - 4t + 1 = 0$

$\Rightarrow \quad (2t-1)^2 = 0 \quad \Rightarrow \quad t=\frac{1}{2} \quad (\because t=\tan\frac{x}{2})$

$\Rightarrow \quad \tan\frac{x}{2}=\frac{1}{2} \quad \Rightarrow \quad \tan\frac{x}{2}=\tan\alpha$, where $\tan\alpha=\frac{1}{2}$

$\Rightarrow \quad \tan\frac{x}{2}=n\pi+a \quad \Rightarrow \quad x=2n\pi+2\alpha$

where $\alpha=\tan^{-1}\left(\frac{1}{2}\right)$, $n \in I$.

CHECK YOUR UNDERSTANDING-I

1. Find the number of solutions of the trigonometric equation $4\cos^2 x + \tan^2 x + \cot^2 x + \sec^2 x = 6$ in $[0, 2\pi]$.
2. Consider the trigonometric equation $\tan x(\sin 2x + 1) = \sin x\,(2 + \tan x)$. Find the number of solutions of the equation in $(0, 4\pi)$.
3. Find real values of x for which, $27^{\cos 2x}\cdot 81^{\sin 2x}$ is minimum. Also find this minimum value.
4. Solve the equation: $2\sin x = 3x^2 + 2x + 3$.
5. Solve: $2 + 7\tan^2\theta = 3.25\sec^2\theta\ (0° < \theta < 360°)$.

Solving Equations with the Use of the Boundness of the Functions sin *x* and cos *x*

EXAMPLE 30

Solve the equation $\dfrac{1-\tan x}{1+\tan x} = 1 + \sin 2x$.

SOLUTION

The equation makes no sense for $x=\frac{\pi}{2}+k\pi$ and for $x=-\frac{\pi}{4}+k\pi$. For all the other values of x it is equivalent to the equation $\dfrac{\cos x-\sin x}{\cos x+\sin x}=1+\sin 2x$.

After simple transformations we obtain $\sin x(3+\sin 2x+\cos 2x)=0$.

It is obvious that the equation $\sin 2x + \cos 2x + 3 = 0$ has no solution, and therefore, the original equation is reduced to the equation $\sin x = 0 \Rightarrow x = k\pi$.

EXAMPLE 31

Solve the equation $(\sin x + \cos x)\sqrt{2} = \tan x + \cot x$.

SOLUTION

Let us transform the equation to the form

$\dfrac{1}{\sqrt{2}}\sin x+\dfrac{1}{\sqrt{2}}\cos x=\dfrac{1}{2\sin x\cos x}$ or $\sin\left(x+\frac{\pi}{4}\right)=\dfrac{1}{\sin 2x}$,

i.e., $\sin\left(x+\frac{\pi}{4}\right)\sin 2x = 1.$ **(1)**

We have $|\sin\alpha| \le 1$, and therefore (1) holds if either $\sin\left(x+\frac{\pi}{4}\right)=1$ and $\sin 2x=-1$ or $\sin\left(x+\frac{\pi}{4}\right)=1$ and $\sin 2x = 1$.

But the first two equations have no roots in common while the second two equations have the common roots $x=\frac{\pi}{4}+2k\pi$. Consequently, the roots of the given equation are $x=\frac{\pi}{4}+2k\pi$.

EXAMPLE 32

Solve the equation $\sin^{2n-1} x + 2\cos^{2n-1} x = 2$, where $n \in N$.

SOLUTION

Obviously no solution is possible if $\frac{\pi}{2} < x < 2\pi$ as LHS < 2.

If $0 < x < \frac{\pi}{2}$, then LHS $= \sin^{2n-1} x + 2\cos^{2n-1} x < \sin^2 x + 2\cos^2 x = 1 + \cos^2 x < 2$ when $n \in N - \{1\}$.

Obviously, a solution exists only when $x = 0$

$\Rightarrow$ The general solution is $x = 2m\pi$, $m \in I$.

When $n = 1$, $\sin x + 2\cos x = 2\sin\frac{x}{2}\left(2\sin\frac{x}{2} - \cos\frac{x}{2}\right) = 0$

$\Rightarrow$ either $x = 2k_1\pi$ or $x = 2k_2\pi + 2\tan^{-1}\frac{1}{2}$, k_1, $k_2 \in I$.

EXAMPLE 33

Solve the equation $\cos^2\left[\frac{\pi}{4}\left(\sin x + \sqrt{2}\cos^2 x\right)\right] - \tan^2\left(x + \frac{\pi}{4}\tan^2 x\right) = 1$.

SOLUTION

$$\cos^2\left[\frac{\pi}{4}\left(\sin x + \sqrt{2}\cos^2 x\right)\right] - \tan^2\left(x + \frac{\pi}{4}\tan^2 x\right) = 1$$

since square of the cosine of any argument doesn't exceed 1, the given equation holds true if and only if we have, simultaneously

$$\cos^2\left[\frac{\pi}{4}\left(\sin x + \sqrt{2}\cos^2 x\right)\right] = 1 \quad \textbf{(1)}$$

and

$$\tan\left(x + \frac{\pi}{4}\tan^2 x\right) = 0 \quad \textbf{(2)}$$

from (1), $\sin x + \sqrt{2}\cos^2 x = 4k$ **(3)**

$\forall k \in I$ but $|\sin x + \sqrt{2}\cos^2 x| \le |\sin x| + \sqrt{2}|\cos^2 x| \le 1 + \sqrt{2} < 4$

so, equation (3) has no solution for $k \ne 0$ for $k = 0$

$\sin x + \sqrt{2}\cos^2 x = 0$ or, $\sqrt{2}\sin^2 x - \sin x - \sqrt{2} = 0$ or,

$\sin x = \frac{-1}{\sqrt{2}}, \sqrt{2}$

but $\sin x = \sqrt{2}$ is not possible. So only solution to the equation (1) is

$$x_1 = \frac{-\pi}{4} + 2n\pi_1,\ x_2 = \frac{5\pi}{4} + 2n\pi_1,\ n = 0, \pm 1, \pm 2\ldots$$

for $x_1 = \frac{-\pi}{4} + 2n\pi$, equation (2) becomes an identity

but $x_2 = \frac{5\pi}{4} + 2n\pi$ doesn't satisfy equation (2)

so, solution to the original equation $x = \frac{-\pi}{4} + 2n\pi\ \forall n \in I$.

EXAMPLE 34

Find the general solution of the equation, $\sin 3x + \cos 4x - 4\sin 7x = \cos 10x + \sin 17x$.

SOLUTION

$(\sin 17x - \sin 3x) + \cos 10x - \cos 4x + 4\sin 7x = 0$

$\Rightarrow$ $2\cos 10x \sin 7x + 2\sin 7x \sin 3x + 4\sin 7x = 0$

$\Rightarrow$ $\sin 7x(\cos 10x - \sin 3x + 2) = 0$
Hence, $\sin 7x = 0$

$\Rightarrow$ $x = \frac{n\pi}{7}$, $n \in I$ or $\cos 10x - \sin 3x + 2 = 0$

$\Rightarrow$ $\cos 10x = -1$ and $\sin 3x = 1$ given $x = (4n + 1)\frac{\pi}{6}$

i.e., $x = -\frac{3\pi}{6}^*, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}^*, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{21\pi}{6}^*, \ldots, \frac{33\pi}{6}\ldots$

Those starred also satisfy $\cos 10x = -1$, the general term of which is

$x = 3(4k - 1)\frac{\pi}{6}$ $k \in I$. Hence, $x = \frac{n\pi}{7}$ or $3(4k - 1)\frac{\pi}{6}$ where $n, k \in I$.

Simultaneous Equations

EXAMPLE 35

Solve the system of equations form

$$\left.\begin{array}{l}\sin x = \csc x + \sin y,\\ \cos x = \sec x + \cos y.\end{array}\right\}$$

SOLUTION

Transform the system to the

$$\left.\begin{array}{l}\sin^2 x = 1 + \sin x \sin y,\\ \cos^2 x = 1 + \cos x \cos y.\end{array}\right\} \quad (1)$$

Adding together the equations of system (1) and subtracting the first equation form the second we obtain the system

$$\left.\begin{array}{l}\cos 2x - \cos(x+y) = 0,\\ 1 + \cos(x+y) = 0.\end{array}\right\} \quad (2)$$

The first equation of system (2) can be rewritten as

$$\cos 2x - \cos (x+y) = 2 \sin\left(\frac{3x+y}{2}\right) \sin (y-x) = 0.$$

If $\sin (x-y) = 0$, then $x - y = k\pi$. But from the second equation of system (2) we find

$$\cos (x-y) = -1, x - y = (2n+1)\pi.$$

Consequently, in this case we have an infinitude of solutions: $x - y = (2n+1)\pi$.

If $\sin\left(\frac{3x+y}{2}\right) = 0$, then $3x = y = 2k\pi$. But $x - y = (2n+1)\pi$

$$\Rightarrow \quad x = \frac{2k+2n+1}{4}\pi, y = \frac{2k-6n-3}{4}\pi.$$

EXAMPLE 36

Solve the system of equations $\sin x \sin y = \frac{\sqrt{3}}{4}$, $\cos x \cos y = \frac{\sqrt{3}}{4}$.

SOLUTION

Adding up the equations of the system, we arrive at an equation

$$\sin x \sin y + \cos x \cos y = \frac{\sqrt{3}}{2} \Leftrightarrow \cos (x-y) = \frac{\sqrt{3}}{2}.$$

Subtracting the first equation of the system from the second. we arrive at an equation

$$\cos x \cos y - \sin x \sin y = 0 \Leftrightarrow \cos (x+y) = 0,$$

Thus, the initial system is equivalent to the system $\cos (x-y) \frac{\sqrt{3}}{2}, x - y = \pm\frac{\pi}{6} + 2\pi n, \Leftrightarrow n, k \in Z, \cos (x+y) = 0, x + y = \frac{\pi}{2} + \pi k$, whence, $x = \frac{\pi}{3} + \frac{\pi}{2}(2n+k)$,

$$x = \frac{\pi}{6} + \frac{\pi}{2}(2n+k),$$

$$y = \frac{\pi}{6} + \frac{\pi}{2}(k-2n), y = \frac{\pi}{3} + \frac{\pi}{2}(k-2n).$$

$$\Rightarrow \quad \frac{\pi}{3} + \frac{\pi}{2}(2n-k), \frac{\pi}{6} + \frac{\pi}{2}(k-2n); \frac{\pi}{6} + \frac{\pi}{2}(2n+k),$$

$$\frac{\pi}{3} + \frac{\pi}{2}(k-2n) \ (k, n \in Z).$$

Miscellaneous Questions

EXAMPLE 37

Solve the equation: $2 \cot 2x - 3 \cot 3x = \tan 2x$.

SOLUTION

The given equation can be rewritten in the form $3\left(\frac{\cos 2x}{\sin 2x} - \frac{\cos 3x}{\sin 3x}\right) = \frac{\sin 2x}{\cos 2x} + \frac{\cos 2x}{\sin 2x}$ or

$$\frac{3 \sin x}{\sin 2x \sin 3x} = \frac{1}{\sin 2x \cos 2x}.$$

Notes

This equation has sense if the condition $\sin 2x \neq 0$, $\sin 3x \neq 0$, $\cos 2x \neq 0$ holds. For the values of x satisfying this condition, we have $3 \sin x \cos 2x = \sin 3x$. Transforming the last equation, we obtain $\sin x (3 - 4 \sin^2 x - 3 \cos 2x) = 0$ and thus, arrive at the equation $2 \sin^3 x = 0$, which is equivalent to the equation $\sin x = 0$. Hence, due to the above note, the original equation has no solutions.

EXAMPLE 38

Solve the equation

$$\tan\left(x-\frac{\pi}{4}\right)\tan x\tan\left(x+\frac{\pi}{4}\right)=\frac{4\cos^2 x}{\tan\frac{x}{2}-\cot\frac{x}{2}}.$$

SOLUTION

The right-hand side (R.H.S.) of the equation is not determined for $x = k\pi$ and $x = \frac{\pi}{2} + m\pi$, because for $x = \frac{2}{\pi}$ the function $\cot\frac{x}{2}$ is not defined, for $x = \left(\frac{2}{\pi}+1\right)\pi$ the function $\tan\frac{x}{2}$ is not defined and for $x = \frac{\pi}{2} + m\pi$ the denominator of the right member of the right member vanishes. For $x \neq k\pi$ we have

$$\tan\frac{x}{2}-\cot\frac{x}{2}=\frac{\sin^2\frac{x}{2}-\cos^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}=-\frac{\cos x}{\sin x}.$$

Hence, for $x \neq k\pi$ and $x \neq \frac{\pi}{2} + m\pi$ (where k and m are arbitrary integers) the right member of the equation is equal to $-2\sin x\cos x$.

The left member of the equation has no sense for $x = \frac{\pi}{2} + k\pi$ and $x = \frac{\pi}{4} + l\cdot\frac{\pi}{2}$ ($l = 0, \pm1, \pm2,\ldots$), and for all the other values of x it is equal to $-\tan x$ because

$$\tan\left(x-\frac{\pi}{4}\right)\tan\left(x+\frac{\pi}{4}\right)=\tan\left(x-\frac{\pi}{4}\right)\cot\left[\frac{\pi}{2}-\left(x+\frac{\pi}{4}\right)\right].$$

$$=-\tan\left(x-\frac{\pi}{4}\right)\cot\left(x-\frac{\pi}{4}\right)=-1$$

Thus, if $x \neq k\pi$, $x \neq \frac{\pi}{2} + m\pi$ and $x \neq \frac{\pi}{4} + l\frac{\pi}{2}$, then the original equation is reduced to the form

$$\tan x = 2\sin x\cos x.$$

This equation has the roots $x = k\pi$ and $x = \frac{\pi}{4} + l\frac{\pi}{2}$.

It follows that the original equation has no roots.

EXAMPLE 39

Solve the equation $(1 + k)\cos x\cos(2x - \alpha) = (1 + k\cos 2x)\cos(x - \alpha)$.

SOLUTION

Let us rewrite the given equation in the form

$$(1 + k)\cos x\cos(2x - \alpha) = \cos(x - \alpha) + k\cos 2x\cos(x - \alpha). \quad \textbf{(1)}$$

We have $\cos x\cos(2x - \alpha) = \frac{1}{2}[\cos(3x - \alpha) + \cos(x - \alpha)]$

and $\cos(x - \alpha)\cos 2x = \frac{1}{2}[\cos(3x - \alpha) + \cos(x + \alpha)]$,

and therefore, equation (1) turns into $k[\cos(x - \alpha) - \cos(x + \alpha)] = \cos(x - \alpha) - \cos(3x - \alpha)$,

that is $\quad k\sin x\sin\alpha = \sin(2x - \alpha)\sin x. \quad$ **(2)**

Equation (2) is equivalent to the following two equations:

(a) $\sin x = 0$; $x = l\pi$ and

(b) $\sin(2x - \alpha) = k\sin\alpha$.

Thus, $x = \frac{\alpha}{2} + (-1)^n, \frac{1}{2}\text{arc}\sin(k\sin\alpha) + \frac{\pi}{2}n$.

For the last expression to make sense, k and α must satisfy the condition $|k\sin\alpha| \leq 1$.

EXAMPLE 40

Solve the equation $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$.

SOLUTION

We designate $t = \tan\frac{x}{2}$ and, using the formulas of the universal trigonometric substitution, write the equation in the form $\frac{3t^4 + 6t^3 + 8t^2 - 2t - 3}{(t^2+1)(1-t^2)} = 0$, its roots are $t_1 = \frac{1}{\sqrt{3}}$, $t_2 = -\frac{1}{\sqrt{3}}$. Thus, the solution of the equation reduces that of two elementary equations

$$\tan\frac{x}{2}=\frac{1}{\sqrt{3}},\ \tan\frac{x}{2}=-\frac{1}{\sqrt{3}} \quad \textbf{(1)}$$

Verification shows that the numbers πn which are roots of the equation $\cos\frac{\pi}{2} = 0$, are not the roots of the given equation, and consequently, all solutions of the initial equation can be found as solutions of equation (1)

$$\Rightarrow \quad x = \pm\frac{\pi}{3} + 2\pi k, \quad (k \in Z).$$

EXAMPLE 41

Solve the equation, $5\sin x + \dfrac{5}{2\sin x} - 5 = 2\sin^2 x + \dfrac{1}{2\sin^2 x}$ if $x \in (0, \pi)$.

SOLUTION

$$5\left(\sin x + \frac{1}{2\sin x}\right) - 5 = 2\left(\sin^2 x + \frac{1}{4\sin^2 x}\right)$$

$$= 2\left(\left(\sin x + \frac{1}{2\sin x}\right)^2 - 1\right)$$

Let $\sin x + \dfrac{1}{2\sin x} = t \quad \Rightarrow \quad 5t - 5 = 2(t^2 - 1)$

$\Rightarrow \quad 2t^2 - 5t + 3 = 0 \quad \Rightarrow \quad (2t - 3)(t - 1) = 0$

$\Rightarrow \quad t = 1 \quad$ or $\quad t = \dfrac{3}{2}$

If $t = 1$, $2\sin^2 x - 2\sin x + 1 = 0 \Rightarrow D < 0$ hence, no solution

If $\quad t = \dfrac{3}{2}$, $2\sin^2 x - 3\sin x + 1 = 0$

$\Rightarrow \quad \sin x = 1 \quad$ or $\quad \sin = \dfrac{1}{2}$

$\therefore \quad x = \dfrac{\pi}{2} \quad$ or $\quad \dfrac{\pi}{6}, \dfrac{5\pi}{6} \quad \Rightarrow \quad x \in \left\{\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}\right\}$.

EXAMPLE 42

Let A and B be acute positive angles satisfying the equalities $3\sin^2 A + 2\sin^2 B = 1$; $3\sin 2A - 2\sin 2B = 0$. Prove that $A + 2B = \dfrac{\pi}{2}$.

SOLUTION

From the given relations, we get $\sin 2B = \dfrac{3}{2}\sin 2A$, $3\sin^2 A = 1 - 2\sin^2 B = \cos 2B$, hence, $\cos(A + 2B) = \cos A \cos 2B - \sin A \sin 2B = \cos A \cdot 3\sin^2 A - \dfrac{3}{2}\sin A \sin 2A = 0$.

CHECK YOUR UNDERSTANDING-II

1. Solve the following system of equation for x and y $5^{\left(\operatorname{cosec}^2 x - 3\sec^2 y\right)} = 1$ and $2^{\left(2\operatorname{cosec} x + \sqrt{3}|\sec y|\right)} = 64$.
2. Find the general solution of the equation, $2 + \tan x \cdot \cot\dfrac{x}{2} + \cot x \cdot \tan\dfrac{x}{2} = 0$.
3. Solve for x, the equation $\sqrt{13 - 8\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.
4. Find all values of θ between 0° & 180° satisfying the equation;
   $$\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0.$$
5. Find the general solution of the following equation: $2(\sin x - \cos 2x) - \sin 2x\,(1 + 2\sin x) + 2\cos x = 0$.
6. Solve: $\tan^2 x \cdot \tan^2 3x \cdot \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$.

PRACTICE EXERCISES

1. What are the most general values of θ which satisfy the equations,
   (a) $\sin\theta = \dfrac{1}{\sqrt{2}}$ (b) $\tan(x - 1) = \sqrt{3}$
   (c) $\tan\theta = -1$ (d) $\operatorname{cosec}\theta = \dfrac{2}{\sqrt{3}}$
   (e) $2\cot^2\theta = \operatorname{cosec}^2\theta$
2. Solve: $\sin 9\theta = \sin\theta$.
3. Solve: $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$.
4. Solve: $\sin 2\theta = \cos 3\theta$.
5. Solve: $\cot\theta = \tan 8\theta$.
6. Solve: $\tan^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 0$.
7. Find all the angles between 0º and 90º which satisfy the equation $\sec^2\theta \cdot \operatorname{cosec}^2\theta + 2\operatorname{cosec}^2\theta = 8$.
8. Solve: $4\cos\theta - 3\sec\theta = 2\tan\theta$.
9. Solve: $\cot\theta - \tan\theta = 2$.
10. Solve: $\sin\theta + \sin 3\theta + \sin 5\theta = 0$.
11. Solve: $\cos\theta + \sin\theta = \cos 2\theta + \sin 2\theta$.
12. Find all values of θ between 0° and 180° satisfying the equation; $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$.
13. Solve: $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.
14. Solve: $\sin^2 n\theta - \sin^2(n - 1)\theta = \sin^2\theta$, where n is constant and $n \neq 0, 1$.

15. Solve: $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

16. Solve: $\operatorname{cosec}\theta = \cot\theta + \sqrt{3}$.

17. Solve: $5\sin\theta + 2\cos\theta = 5$.

18. Solve: $\tan 2\theta \tan\theta = 1$.

19. Solve: $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.

20. Solve: $\tan x\cdot\tan\left(x+\frac{\pi}{3}\right)\cdot\tan\left(x+\frac{2\pi}{3}\right) = \sqrt{3}$.

21. If $\tan\theta + \sin\phi = \frac{3}{2}$ $\tan^2\theta + \cos^2\phi = \frac{7}{4}$ then find the general value of θ and ϕ.

22. Solve the equation for $0 \le \theta \le 2\pi$; $(\sin 2\theta + \sqrt{3}\cos 2\theta)^2 - 5 = \left(\frac{\pi}{6} - 2\theta\right)$.

23. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2}+\log_5(\sin x)} = 15^{\frac{1}{2}+\log_{15}\cos x}$.

24. Find all the values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

25. Solve the equality: $2\sin 11x + \cos 3x + \sqrt{3}\sin 3x = 0$.

26. Find all value of θ, between 0 and π, which satisfy the equation; $\cos\theta\cdot\cos 2\theta\cdot\cos 3\theta = \frac{1}{4}$.

27. Given that A, B are positive acute angle, solve: $\sqrt{3}\sin^2 2A = \sin 2B$ and $\sqrt{3}\sin^2 A + \sin^2 B = \frac{\sqrt{3}-1}{2}$.

28. Find the general solution of the equation, $\tan^2(x+y) + \cot^2(x+y) = 1 - 2x - x^2$.

29. Prove that the equations
(a) $\sin x\cdot\sin 2x\cdot\sin 3x = 1$
(b) $\sin x\cdot\cos 4x\cdot\sin 5x = -\frac{1}{2}$ have no solution

30. Find all the solutions of, $4\cos^2 x\sin x - 2\sin^2 x = 3\sin x$.

31. Solve the inequality $\sin 2x > \sqrt{2}\sin^2 x + (2-\sqrt{2})\cos^2 x$.

32. Find the set of values of 'a' for which the equation, $\sin^4 x + \cos^4 x + \sin 2x + a = 0$ possesses solutions. Also find the general solution for these values of 'a'.

33. Find the set of values of x satisfying the equality $\sin\left(x - \frac{\pi}{4}\right) - \cos\left(x + \frac{3\pi}{4}\right) = 1$ and the inequality $\frac{2\cos 7x}{\cos 3 + \sin 3} > 2^{\cos 2x}$.

34. If 'S' is the solution set in $[0, \pi/2]$ of the trigonometrical equation $\cos^6 x + \sin^6 x - \cos^2 2x = \frac{1}{16}$. Then the set '$S$' is __________.

35. Solve the inequality $\sin 3x < \sin x$.

36. Find all values of 'a' for which every root of the equation, $a\cos 2x + |a|\cos 4x + \cos 6x = 1$ is also a root of the equation, $\sin x\cos 2x = \sin 2x\cos 3x - \frac{1}{2}\sin 5x$, and conversely, every root of the second equation is also a root of the first equation.

37. Solve the equality: $2\sin 11x + \cos 3x + \sqrt{3}\sin 3x = 0$

38. Find the general solution of the equation, $2 + \tan x\cdot\cot\frac{x}{2} + \cot x\cdot\tan\frac{x}{2} = 0$

39. Solve for x, the equation $\sqrt{13 - 18\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.

40. Solve the equation for x,

$$2\sin\left(3x + \frac{\pi}{4}\right) = \sqrt{1 + 8\sin 2x\cdot\cos^2 2x}$$

ASSESSMENT

JEE Main

Level I

1. The general solution of the equation, $2\cos 2x = 3.\ 2\cos^2 x - 4$ is
(A) $x = 2n\pi, n \in I$ (B) $x = n\pi, n \in I$
(C) $x = \frac{n\pi}{4}, n \in I$ (D) $x = \frac{n\pi}{2}, n \in I$

2. The solution set of the equation $4\sin\theta\cdot\cos\theta - 2\cos\theta - 2\sqrt{3}\sin\theta + \sqrt{3} = 0$ in the interval $(0, 2\pi)$ is
(A) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ (B) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$
(C) $\left\{\frac{3\pi}{4}, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (D) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}\right\}$

3. Total number of solutions of $\sin x\cdot\tan 4x = \cos x$ belonging to $(0, \pi)$ are
(A) 4 (B) 7
(C) 8 (D) 5

4. If $2\cos^2(\pi + x) + 3\sin(\pi + x)$ vanishes then the values of x lying in the interval from 0 to 2π are

(A) $y = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ (B) $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

(C) $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ (D) $x = \frac{\pi}{2}$ or $\frac{5\pi}{2}$

5. If $x \in \left[0, \frac{\pi}{2}\right]$, the number of solutions of the equation, $\sin 7x + \sin 4x + \sin x = 0$ is

(A) 3 (B) 5

(C) 6 (D) None of these

6. The general solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$ is

(A) $2n\pi;\ n \in I$ (B) $n\pi;\ n \in I$

(C) $\frac{n\pi}{3};\ n \in I$ (D) $2\frac{n\pi}{3};\ n \in I$

7. $\frac{\cos 3\theta}{2\cos 2\theta - 1} = \frac{1}{2}$ if

(A) $\theta = n\pi + \frac{\pi}{3},\ n \in I$

(B) $\theta = 2n\pi \pm \frac{\pi}{3},\ n \in I$

(C) $\theta = 2n\pi \pm \frac{\pi}{6},\ n \in I$

(D) $\theta = n\pi + \frac{\pi}{6},\ n \in I$

8. If $\cos 2\theta + 3\cos\theta = 0$ then

(A) $\theta = 2n\pi \pm \alpha$, where $\alpha = \cos^{-1}\left(\frac{\sqrt{17} - 3}{4}\right)$

(B) $\theta = 2n\pi \pm \alpha$, where $\alpha = \cos^{-1}\left(\frac{-\sqrt{17} - 3}{4}\right)$

(C) $\theta = 2n\pi \pm \alpha$, where $\alpha = \cos^{-1}\left(\frac{\pm\sqrt{17} - 3}{4}\right)$

(D) None of these

9. If $\sin\theta + 7\cos\theta = 5$, then $\tan\left(\frac{\theta}{2}\right)$ is a root of the equation

(A) $x^2 - 6x + 1 = 0$ (B) $6x^2 - x - 1 = 0$

(C) $6x^2 + x + 1 = 0$ (D) $x^2 - x + 6 = 0$

10. $\sin 3\theta = 4\sin\theta \cdot \sin 2\theta \cdot \sin 4\theta$ in $0 \le \theta \le \pi$ has

(A) 2 real solutions (B) 4 real solutions

(C) 6 real solutions (D) 8 real solutions

11. General solution of the equation, $\cot 3\theta - \cot\theta = 0$ is

(A) $\theta = (2n - 1)\frac{\pi}{2},\ n \in I$

(B) $\theta = (2n - 1)\frac{\pi}{4},\ n \in I$

(C) $\theta = (2n - 1)\frac{\pi}{3},\ n \in I$

(D) None of these

12. The principal solution set of the equation, $2\cos x = \sqrt{2 + 2\sin 2x}$ is

(A) $\left\{\frac{\pi}{8}, \frac{13\pi}{8}\right\}$ (B) $\left\{\frac{\pi}{4}, \frac{13\pi}{8}\right\}$

(C) $\left\{\frac{\pi}{4}, \frac{13\pi}{10}\right\}$ (D) $\left\{\frac{\pi}{8}, \frac{13\pi}{10}\right\}$

13. The number of all possible triplets (a_1, a_2, a_3) such that: $a_1 + a_2\cos 2x + a_3\sin^2 x = 0$ for all x is

(A) 0 (B) 1

(C) 2 (D) infinite

14. The arithmetic mean of the roots of the equation $4\cos^3 x - 4\cos^2 x - \cos(\pi + x) - 1 = 0$ in the interval [0, 315] is equal to

(A) 49π (B) 50π

(C) 51π (D) 100π

15. The general solution of $\tan\left(\frac{2}{3}\theta\right) = \sqrt{3}$ is

(A) $\frac{3n\pi}{2} + \frac{\pi}{2};\ n \in I$

(B) $\frac{n\pi}{2};\ \pm\frac{\pi}{2};\ n \in I$

(C) $n\pi \pm \frac{\pi}{2};\ n \in I$

(D) None of these

16. Find the general value of θ, when $\cos\left(\frac{-\theta}{2}\right) = 0$

(A) $(n + 1)\pi;\ n \in I$

(B) $n\pi;\ n \in I$

(C) $(2n + 1)\pi;\ n \in I$

(D) $2n\pi;\ n \in I$

17. If $\tan a\theta - \tan b\theta = 0$, then the values of θ form a series in

(A) A.P. (B) G.P.
(C) H.P. (D) None of these

18. The general solution of equation $\sin^2\theta \sec\theta + \sqrt{3}\tan\theta = 0$ is

(A) $\theta = n\pi + (-1)^{n+1}\dfrac{\pi}{3}$
(B) $\theta = n\pi$
(C) $\theta = n\pi + (-1)^{n+1}\dfrac{\pi}{6}$
(D) $\theta = \dfrac{n\pi}{2}$

19. If $(1 + \tan\theta)(1 + \tan\phi) = 2$, then $\theta + \phi =$

(A) 30° (B) 45°
(C) 60° (D) 75°

20. The general solution of $\sin x + 3\sin 2x + \sin 3x = \cos x + 3\cos 2x + \cos 3x$ then in the interval $0 \le x \le 2\pi$, $x =$

(A) $\dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{2\pi}{3}$
(B) $\dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$
(C) $\dfrac{4\pi}{3}, \dfrac{9\pi}{3}, \dfrac{2\pi}{3}, \dfrac{13\pi}{8}$
(D) $\dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{3}, \dfrac{4\pi}{3}$

21. The general value of θ satisfying the equation $\sin^2\theta - 2\cos\theta + \dfrac{1}{4} = 0$

(A) $2n\pi \pm \dfrac{\pi}{3}$ (B) $2n\pi \pm \dfrac{\pi}{4}$
(C) $2n\pi \pm \dfrac{\pi}{6}$ (D) None of these

22. If $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta\tan 4\theta\tan 7\theta$, then $\theta =$

(A) $\dfrac{n\pi}{4}$
(B) $\dfrac{n\pi}{7}$
(C) $\dfrac{n\pi}{12}$; $n \ne 6(2k+1)$
(D) $n\pi$

23. If $3(\sec^2\theta + \tan^2\theta) = 5$, then the general value of θ is

(A) $2n\pi + \dfrac{\pi}{6}$ (B) $2n\pi \pm \dfrac{\pi}{6}$
(C) $n\pi \pm \dfrac{\pi}{6}$ (D) $n\pi \pm \dfrac{\pi}{3}$

24. Smallest positive x satisfying the equation $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cdot \cos^3 x$ is

(A) 15° (B) 18°
(C) 22.5° (D) 30°

25. The solutions of the equation, $(1+\cos x)\sqrt{\tan\dfrac{x}{2}} - 2 + \sin x = 2\cos x$ are identical to the solutions of the equation

(A) $\sin x = 1$ (B) $\cos x = 0$
(C) $\sin 2x = 0$ (D) $\sec(x/2) = \sqrt{2}$

26. $\sin 3\theta = 4\sin\theta\sin 2\theta\sin 4\theta$ in $0 \le \theta \le \pi$ has

(A) 2 real solutions (B) 4 real solutions
(C) 6 real solutions (D) 8 real solutions.

27. The equation, $\sin^2 x - \dfrac{4}{\sin^3 x - 1} = 1 - \dfrac{4}{\sin^3 x - 1}$ has

(A) no root (B) one root
(C) two roots (D) infinite roots

28. The number of solutions of $\tan(5\pi\cos\theta) = \cot(5\pi\sin\theta)$ for θ in $(0, 2\pi)$ is:

(A) 28 (B) 14
(C) 4 (D) 2

29. The value of x satisfying the equation $\sqrt{1+\sin x} - \sqrt{1-\sin x} = 2\sin\dfrac{x}{2}$ are where $n \in I$

(A) $2n\pi < x < 2(n+1)\pi$
(B) $2n\pi - \pi/2 < x < 2n\pi + \pi/2$
(C) $n\pi - \pi/4 < x < n\pi + \pi/4$
(D) $(2n-1)\pi/2 < x < (2n+1)\pi$

30. If $x = \dfrac{n\pi}{2}$, satisfies the equation $\sin\dfrac{x}{2} - \cos\dfrac{x}{2} = 1 - \sin x$ and the inequality $\left|\dfrac{x}{2} - \dfrac{x}{2}\right| \le \dfrac{3\pi}{4}$, then:

(A) $n = -1, 0, 3, 5$ (B) $n = 1, 2, 4, 5$
(C) $n = 0, 2, 4$ (D) $n = -1, 1, 3, 5$

31. There is a formula that says that

$$\sin 7x = A\sin^7 x + B\sin^6 x + C\sin^5 x + D\sin^4 x + E\sin^3 x + F\sin^2 x + G\sin x + H.$$

The value of the sum $(A + B + C + D + E + F + G + H)$, is

(A) 0 (B) 1
(C) –1 (D) not possible to determine

32. Number of solutions of the equation

$\cos^2 x+\left(\frac{\sqrt{3}+1}{2}\right)\sin x=\frac{\sqrt{3}}{4}+1$ in the interval $[-\pi, \pi]$, is

(A) 2 (B) 4
(C) 6 (D) 8

33. The expression $\sin x - \cos^2 x - 1$ assumes the least value for the set of real values of x given by

(A) $x=n\pi+(-1)^{n+1}\left(\frac{\pi}{6}\right)$

(B) $x=n\pi+(-1)^{n}\left(\frac{\pi}{6}\right)$

(C) $x=n\pi+(-1)^{n}\left(\frac{\pi}{3}\right)$

(D) $x=n\pi-(-1)^{n}\left(\frac{\pi}{3}\right)$

where $n \in$ I

34. The number of roots of the equation, $\sin x + 2\sin 2x = 3 + \sin 3x$ is

(A) 0 (B) 1
(C) 2 (D) infinite

35. The smallest positive root of the equation $\sqrt{\sin(1-x)}=\sqrt{\cos x}$ is

(A) $\frac{3\pi}{4}+\frac{1}{2}$ (B) $\frac{7\pi}{4}+\frac{1}{2}$

(C) $\frac{11\pi}{4}+\frac{1}{2}$ (D) $\frac{\pi}{4}+\frac{1}{2}$

Level II

1. All solutions of the equation, $2\sin\theta+\tan\theta=0$ are obtained by taking all integral values of m and n in

(A) $2n\pi+\frac{2\pi}{3}, n\in I$

(B) $n\pi$ or $2m\pi\pm\frac{2\pi}{3}$ where $n, m\in I$

(C) $n\pi$ or $m\pi\pm\frac{\pi}{3}$ where $n, m\in I$

(D) $n\pi$ or $2m\pi\pm\frac{\pi}{3}$ where $n, m\in I$

2. The most general solution of $\tan\theta=-1$ and $\cos\theta=\frac{1}{\sqrt{2}}$ is

(A) $n\pi+\frac{7\pi}{4}, n\in I$ (B) $n\pi+(-1)^n\frac{7\pi}{4}, n\in I$

(C) $2n\pi+\frac{7\pi}{4}, n\in I$ (D) None of these

3. A triangle ABC is such that $\sin(2A+B)=\frac{1}{2}$. If A, B, C are in A.P. then the angle A, B, C are, respectively

(A) $\frac{5\pi}{12},\frac{\pi}{4},\frac{\pi}{3}$ (B) $\frac{\pi}{4},\frac{\pi}{3},\frac{5\pi}{12}$

(C) $\frac{\pi}{3},\frac{\pi}{4},\frac{5\pi}{12}$ (D) $\frac{\pi}{3},\frac{5\pi}{12},\frac{\pi}{4}$

4. $\frac{\sin 3\theta}{2\cos 2\theta+1}=\frac{1}{2}$ if

(A) $\theta=n\pi+\frac{\pi}{6}, n\in I$

(B) $\theta=2n\pi-\frac{\pi}{6}, n\in I$

(C) $\theta=n\pi+(-1)^n\frac{\pi}{6}, n\in I$

(D) $\theta=n\pi-\frac{\pi}{6}, n\in I$

5. The set of values of x for which $\frac{\tan 3x-\tan 2x}{1+\tan 3x\tan 2x}=1$ is

(A) ϕ

(B) $\left(\frac{\pi}{4}\right)$

(C) $\{n\pi+\frac{\pi}{4} \mid n=1, 2, 3 \ldots.\}$

(D) $\{2n+\frac{\pi}{4} \mid n=1, 2, 3 \ldots.\}$

6. The value 'a' for which the equation $4\,\text{cosec}^2(\pi(a+x))+a^2-4a=0$ has a real solution is:

(A) $a=1$ (B) $a=2$
(C) $a=3$ (D) None of these

7. The solution of $|\cos x| = \cos x - 2\sin x$ is

(A) $x=n\pi, n\in I$

(B) $x=n\pi+\frac{\pi}{4}, n\in I$

(C) $x=n\pi+(-1)^n\frac{\pi}{4}, n\in I$

(D) $(2n+1)\pi+\frac{\pi}{4}, n\in I$

8. The number of solutions of $\sin\theta + 2\sin 2\theta + 3\sin 3\theta + 4\sin 4\theta = 10$ in $(0, \pi)$ is
(A) 1 (B) 2
(C) 4 (D) 0

9. Total number of ordered pairs (x, y) of real numbers satisfying $|\tan \pi y| + (\sin \pi x)^2 = 0$ and $x^2 + y^2 \le 2$ is equal to
(A) 3 (B) 6
(C) 9 (D) 12

10. The solution of the equation $5^{(\sin 2x - \tan x)} - 4^{(5\cos 2x - \tan x)} = 5^{(1 - \tan x)} (0.04)^{\sin^2 x}$ is
(A) $(2n+1)\frac{\pi}{2}; m\pi + \frac{\pi}{4}$
(B) $m\pi + \frac{\pi}{4}$
(C) $(2n+1)\frac{\pi}{4}$
(D) all of these
where $m, n \in I$

11. If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 4$, then the most general values of θ are given by
(A) $\theta = n\pi \pm \frac{\pi}{3}$ (B) $\theta = 2n\pi \pm \frac{\pi}{4}$
(C) $\theta = n\pi \pm \frac{\pi}{6}$ (D) $\theta = 2n\pi \pm \frac{\pi}{6}$
where $n \in I$

12. General solution of the equation $\sec x = 1 + \cos x + \cos^2 x + \cos^3 x + \ldots\ldots\ldots \infty$, is
(A) $n\pi + \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{3}$
(C) $n\pi \pm \frac{\pi}{6}$ (D) $2n\pi \pm \frac{\pi}{6}$
where n is an integer.

13. Let

$$f(x) = \frac{\operatorname{cosec}^4 x - 2\operatorname{cosec}^2 x + 1}{\operatorname{cosec} x(\operatorname{cosec} x - \sin x) + \frac{\sin x - \cos x}{\sin x} + \cot x}.$$

The sum of all the solutions of $f(x) = 0$ in $[0, 100\pi]$ is
(A) 2550π (B) 2500π
(C) 5000π (D) 5050π

14. If $32\tan^8\theta = 2\cos^2\alpha - 3\cos\alpha$ and $3\cos 2\theta = 1$, then the most general values of α are
(A) $n\pi - \frac{\pi}{3}$ (B) $2n\pi \pm \frac{2\pi}{3}$
(C) $n\pi + \frac{\pi}{3}$ (D) $n\pi \pm \frac{\pi}{3}$
where $n \in I$

15. Let X be the set of all solutions to the equation $\cos x \cdot \sin\left(x + \frac{1}{x}\right) = 0$. Number of real numbers contained by X in the interval $(0 < x < \pi)$, is
(A) 0 (B) 1
(C) 2 (D) more than 2

16. If $\tan^2 x = \frac{\sqrt{3}-1}{5+3\sqrt{3}}$, then the general values of x are
(A) $n\pi \pm \frac{\pi}{12}$ (B) $n\pi \pm \frac{\pi}{8}$
(C) $n\pi \pm \frac{5\pi}{12}$ (D) $n\pi \pm \frac{3\pi}{8}$
where $n \in I$

17. The general values of x satisfying simultaneously the equations $2\sin 2x + \sqrt{3} = 0$ and $\sqrt{3}\tan x + 1 = 0$ is given by
(A) $2n\pi + \frac{5\pi}{6}, n \in I$
(B) $2n\pi + \frac{\pi}{6}, n \in I$
(C) $n\pi + \frac{5\pi}{6}, n \in I$
(D) $n\pi + \frac{\pi}{6}, n \in I$

18. The true solution set of inequality $\log_2(\sin\theta) > \log_2(\cos\theta)$ is equal to
(A) $\bigcup_{n\in I}\left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{5\pi}{4}\right)$
(B) $\bigcup_{n\in I}\left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{\pi}{2}\right)$
(C) $\bigcup_{n\in I}\left(2n\pi, 2n\pi + \frac{\pi}{4}\right)$
(D) $\bigcup_{n\in I}\left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{7\pi}{4}\right)$

19. In ΔABC, if $\cos A + \sin A - \dfrac{2}{\cos B + \sin B} = 0$, then $\dfrac{a+b}{c}$ is equal to

(A) $\sqrt{2}$ (B) 1

(C) $\dfrac{1}{\sqrt{2}}$ (D) $2\sqrt{2}$

20. If $0 \le \theta \le 2\pi$, then the solution set of the equation $\sin\theta - \sec\theta \cot^3\theta - \text{cosec}\,\theta = 0$, is

(A) ϕ (B) $\{0\}$

(C) $\left\{\dfrac{\pi}{2}\right\}$ (D) $\left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right\}$

Integer-type Questions

1. Find the principal solution of the trigonometric equation

$$\sqrt{\cot 3x + \sin^2 x - \frac{1}{4}} + \sqrt{\sqrt{3}\cos x + \sin x - 2} = \sin\frac{3x}{2} - \frac{\sqrt{2}}{2}.$$

2. Determine the smallest positive value of x which satisfy the equation, $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$.

3. Find the number of solutions of the equation $\sin 2x + \cos 2x + 4\sin x = 1 + 4\cos x$ lying in interval $[-6, 6]$.

4. Find the number of solutions of the equation $\prod_{r=1}^{12} \sin rx = 0$ lying in the interval $(0, \pi]$.

5. Sum of all the solutions in $[0, 4\pi]$ of the equation $\tan x + \cot x + 1 = \cos\left(x + \dfrac{\pi}{4}\right)$ is $k\pi$ then the value of k is

6. If the quadratic equation $x^2 + (2 - \tan\theta)x - (1 + \tan\theta) = 0$ has two integral roots, then sum of all possible values of θ in interval $(0, 2\pi)$ is $k\pi$. Find the value of k.

7. If the values of y satisfying the equation $x^2 - 2x\sin(xy) + 1 = 0$ is expressed in the form of $k\pi$ $(k \in R)$ then find the sum of all possible values of k in $(0, 48)$.

8. Find the number of ordered pairs (x, y) satisfying the equations $\sin x \cos y = 1$ and $x^2 + y^2 \le 9\pi^2$.

9. Find the number of solutions of the equation $\sin^2 2x - \cos^2 8x = \dfrac{1}{2}\cos 10x$ lying in the interval $\left(0, \dfrac{\pi}{2}\right)$.

JEE Advanced

Single Correct Option-type Questions

1. If $20\sin^2\theta + 21\cos\theta - 24 = 0$ and $\dfrac{7\pi}{4} < \theta < 2\pi$ then the values of $\cot\dfrac{\theta}{2}$ is

(A) 3 (B) $\dfrac{\sqrt{15}}{3}$

(C) $-\dfrac{\sqrt{15}}{3}$ (D) -3

2. The general solution of the equation $\tan x + \tan\left(x + \dfrac{\pi}{3}\right) + \tan\left(x + \dfrac{2\pi}{3}\right) = 3$ is

(A) $\dfrac{n\pi}{4} + \dfrac{\pi}{12}, n \in I$ (B) $\dfrac{n\pi}{3} + \dfrac{\pi}{6}, n \in I$

(C) $\dfrac{n\pi}{3} + \dfrac{\pi}{12}, n \in I$ (D) None of these

3. The general solution of the equation $\tan^2\alpha + 2\sqrt{3}\tan\alpha = 1$ is given by

(A) $\alpha = \dfrac{n\pi}{2}, n \in I$

(B) $\alpha = (2n+1)\dfrac{\pi}{2}, n \in I$

(C) $\alpha = (6n+1)\dfrac{\pi}{12}, n \in I$

(D) $\alpha = \dfrac{n\pi}{12}, n \in I$

4. The number of integral values of a for which the equation $\cos 2x + a\sin x = 2a - 7$ possesses a solution is

(A) 2 (B) 3

(C) 4 (D) 5

5. The number of solution of the equation $|\sin x| = |\cos 3x|$ in $[-2\pi, 2\pi]$ is

(A) 32 (B) 28

(C) 24 (D) 30

6. If $2\tan^2 x - 5\sec x - 1 = 0$ has 7 different roots in $\left[0, \frac{n\pi}{2}\right]$, $n \in N$, then greatest value of n is

(A) 8 (B) 10
(C) 13 (D) 15

7. The values of x between 0 and 2π which satisfy the equation $\sin x \cdot \sqrt{8\cos^2 x} = 1$ are in A.P. The common difference of the A.P. is

(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$
(C) $\frac{3\pi}{8}$ (D) $\frac{3\pi}{4}$

8. Number of solution of the equation

$$\frac{3\sin x - \sin 3x}{1+\cos\theta} + \frac{3\cos x + \cos 3x}{1-\sin\theta} = 4\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$$

in the interval $(-10\pi, 8\pi]$ is equal to

(A) 8 (B) 9
(C) 16 (D) 18

9. If the system of equations $x - y = 2$ and $cx + y = 3$ have a solution (x, y) where $x > 0$ and $y > 0$, then the true set of values of c is

(A) $c > -1$ (B) $c < \frac{3}{2}$
(C) $0 < c < \frac{3}{2}$ (D) $-1 < c < \frac{3}{2}$

10. The number of ordered pair(s) (x, y) of real numbers satisfying the equation $1 + x^2 + 2x\sin(\cos^{-1} y) = 0$, is

(A) 0 (B) 1
(C) 2 (D) 3

11. Number of solution of the equation $2\sqrt{\cos^6\theta - \sin^6\theta} = 1 + \cos 2\theta$ in $[-\pi, \pi]$ is

(A) 1 (B) 2
(C) 3 (D) 4

12. The true solution set of the inequality $\sin x + \sin 2x \geq 0$ in $(0, 2\pi)$ is

(A) $\left(0, \frac{2\pi}{3}\right] \cup \left[\pi, \frac{4\pi}{3}\right]$

(B) $\left(0, \frac{\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right)$

(C) $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right)$

(D) $[0, \pi] \cup \left[\frac{5\pi}{3}, 2\pi\right)$

13. The values of x for which min.$(\sin x, \cos x) >$ min.$(-\sin x, -\cos x)$ where $x \in (0, 2\pi)$ equals

(A) $(0, \pi)$ (B) $\left(\frac{3\pi}{4}, 2\pi\right)$
(C) $\left(0, \frac{5\pi}{6}\right)$ (D) $\left(\frac{7\pi}{4}, 2\pi\right) \cup \left(0, \frac{3\pi}{4}\right)$

14. General solution of the equation $|\cos x| = \sin x$, is

(A) $n\pi + (-1)^n \frac{\pi}{4}$ (B) $2n\pi \pm \frac{\pi}{4}$
(C) $n\pi + \frac{\pi}{4}$ (D) $2n\pi + \frac{\pi}{4}$

where $n \in I$

15. Number of solutions of the equation $2\tan^2 x - 5\sec x = 1$ in $(-\pi, 2\pi)$ is equal to

(A) 2 (B) 3
(C) 4 (D) 5

16. General solution of the trigonometric equation,

$$\left(2\sin^4\frac{x}{2} - 1\right)\frac{1}{\cos^4\frac{x}{2}} = 2 \text{ is}$$

(A) $n\pi + \frac{2\pi}{3}$ (B) $n\pi \pm \frac{\pi}{3}$
(C) $2n\pi + \frac{2\pi}{3}$ (D) $2n\pi \pm \frac{2\pi}{3}$

17. Let ABC be an acute triangle with all sides unequal. D, E and F are the feet of the perpendiculars from the vertex A, B and C respectively. The circumcircle of the triangle AEF passes through the

(A) centriod of the ΔABC.
(B) orthocentre of the ΔABC.
(C) incentre of the ΔABC.
(D) circumcentre of the ΔABC.

18. Number of ordered pairs (x, y) of real numbers satisfying the system of equations $\sin x = \sin 2y$ and $\cos x = \sin y$ where $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$, is

(A) 1 (B) 2
(C) 3 (D) 4

19. The true set of values of $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x - 1| < \sqrt{5}$ is equal to

(A) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (B) $\left(\frac{-\pi}{10}, \frac{3\pi}{10}\right)$

(C) $\left(\frac{-\pi}{5}, \frac{3\pi}{5}\right)$ (D) $\left(\frac{-2\pi}{3}, \frac{\pi}{3}\right)$

20. If $2^{\left(\sqrt{\sin^2 x - 2\sin x + 5} - 2\sin^2 y\right)} \le 1$ then number of ordered pairs (x, y) in $[0, 2\pi]$

(A) 0 (B) 1

(C) 2 (D) 3

21. Complete set of values of 'a' for which $a \sin^2 x + |\cos x| - 2a = 0$ has atleast one solution, is

(A) $\left[\frac{-1}{2}, 0\right)$ (B) $\left[0, \frac{1}{2}\right]$

(C) $\left(0, \frac{1}{2}\right]$ (D) $\left(0, \frac{1}{2}\right)$

22. If the equation $p \sin 2x + \cos x = p$ has a solution, then the range of p is

(A) $(-\infty, \infty)$ (B) $[-1, 1]$

(C) $(-\infty, 0]$ (D) $[0, \infty)$

23. Given $0 \le \theta \le 2\pi$, the number of possible solutions satisfying the equation $8\cos^6 2\theta - 7\cos^3 2\theta - 1 = 0$, is

(A) 5 (B) 6

(C) 7 (D) 8

24. If the expression $E = \sin\frac{\theta}{2} - \sqrt{3}\cos\frac{\theta}{2} (\theta \in R)$ attains its maximum value then the most general values of θ are given by

(A) $4k\pi + \frac{5\pi}{3}$ (B) $2k\pi + \frac{5\pi}{3}$

(C) $4k\pi + \frac{5\pi}{6}$ (D) $2k\pi + \frac{5\pi}{6}$

(where $k \in I$)

25. Sum of all the solutions of the equation $\cos 2x + \sin^2 2x = 1$, which lie in the interval $[0, 2\pi]$ is equal to

(A) 4π (B) 5π

(C) 7π (D) 8π

26. Given $\frac{\cos^2\theta - \sin^2\theta}{2\cos^2\theta\tan\theta} = 2 - \sqrt{3}$ then the value of $\theta \in (0, \pi/2)$ is

(A) $\frac{\pi}{24}$ (B) $\frac{5\pi}{12}$

(C) $\frac{5\pi}{24}$ (D) $\frac{\pi}{12}$

27. Let $\theta \in [0, 4\pi]$ satisfying the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all value of θ is of the form $k\pi$ then the value of 'k', is

(A) 6 (B) 5

(C) 4 (D) 2

28. Number of solution of the equation, $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$ in $0 \le x \le 3\pi$, is

(A) 3 (B) 4

(C) 5 (D) 6

Multiple Correct Option-type Questions

1. $\sin x$, $\sin 2x$, $\sin 3x$ are in A.P. if

(A) $x = \frac{n\pi}{2}, n \in I$ (B) $x = n\pi, n \in I$

(C) $x = 2n\pi, n \in I$ (D) $x = (2n+1)\pi, n \in I$

2. $\sin x - \cos^2 x - 1$ assumes the least value for the set of values of x given by

(A) $x = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right), n \in I$

(B) $x = n\pi + (-1)^{n}\left(\frac{\pi}{6}\right), n \in I$

(C) $x = n\pi + (-1)^{n}\left(\frac{\pi}{3}\right), n \in I$

(D) $x = n\pi - (-1)^{n}\left(\frac{\pi}{6}\right), n \in I$

3. $\sin x + \sin 2x + \sin 3x = 0$ if

(A) $\sin x = \frac{1}{2}$ (B) $\sin 2x = 0$

(C) $\sin 3x = \frac{\sqrt{3}}{2}$ (D) $\cos x = -\frac{1}{2}$

4. $\cos 4x \cos 8x - \cos 5x \cos 9x = 0$ if

(A) $\cos 12x = \cos 14x$

(B) $\sin 13x = 0$

(C) $\sin x = 0$

(D) $\cos x = 0$

5. The general solution of the equation $\cos x \cdot \cos 6x = -1$, is
(A) $x = (2n + 1)\pi, n \in I$
(B) $x = 2n\pi, n \in I$
(C) $x = (2n - 1)\pi, n \in I$
(D) None of these

6. If $\sin(x - y) = \cos(x + y) = \frac{1}{2}$ then the values of x and y lying between 0 and π are given by
(A) $x = \frac{\pi}{4}, y = \frac{3\pi}{4}$ (B) $x = \frac{\pi}{4}, y = \frac{\pi}{12}$
(C) $x = \frac{5\pi}{4}, y = \frac{5\pi}{12}$ (D) $x = \frac{11\pi}{12}, y = \frac{3\pi}{4}$

7. The equation $2\sin\frac{x}{2}\cdot\cos^2 x + \sin^2 x = 2\sin\frac{x}{2}\cdot\sin^2 x + \cos^2 x$ has a root for which
(A) $\sin 2x = 1$ (B) $\sin 2x = -1$
(C) $\cos x = \frac{1}{2}$ (D) $\cos 2x = -\frac{1}{2}$

8. $\cos 15x = \sin 5x$ if
(A) $x = -\frac{\pi}{20} + \frac{n\pi}{5}, n \in I$
(B) $x = \frac{\pi}{40} + \frac{n\pi}{10}, n \in I$
(C) $x = \frac{3\pi}{20} + \frac{n\pi}{5}, n \in I$
(D) $x = -\frac{3\pi}{40} + \frac{n\pi}{10}, n \in I$

9. $5\sin^2 x + \sqrt{3}\sin x\cos x + 6\cos^2 x = 5$ if
(A) $\tan x = \frac{-1}{\sqrt{3}}$
(B) $\sin x = 0$
(C) $x = n\pi + \frac{\pi}{2}, n \in I$
(D) $x = n\pi + \frac{\pi}{6}, n \in I$

10. $\sin^2 x + 2\sin x\cos x - 3\cos^2 x = 0$ if
(A) $\tan x = 3$
(B) $\tan x = -1$
(C) $x = n\pi + \frac{\pi}{4}, n \in I$
(D) $x = n\pi + \tan^{-1}(-3), n \in I$

11. $\sin^2 x - \cos 2x = 2 - \sin 2x$ if
(A) $x = \frac{n\pi}{2}$, n $\in I$
(B) $\tan x = \frac{3}{2}$
(C) $x = (2n + 1)\frac{\pi}{2}, n \in I$
(D) $x = n\pi + (-1)^n \sin^{-1}\left(\frac{2}{3}\right), n \in I$

12. Which of the following equation(s) have NO solution?
(A) $\sin x + \sin 9x = 2$
(B) $\cos x - \sin 3x = -2$
(C) $\sin\left(\frac{5x}{2}\right) - \sin\left(\frac{x}{2}\right) = 2$
(D) $\cos x \cos 6x = -1$

13. Let $2\sin x + 3\cos y = 3$ and $3\sin y + 2\cos x = 4$ then
(A) $x + y = (4n + 1)\pi/2, n \in I$
(B) $x + y = (2n + 1)\pi/2, n \in I$
(C) x and y can be the two non right angles of a 3-4-5 triangle with $x > y$.
(D) x and y can be the two non right angles of a 3-4-5 triangle with $y > x$.

14. The equation $\operatorname{cosec} x + \sec x = 2\sqrt{2}$ has
(A) no solution in $\left(0, \frac{\pi}{4}\right)$
(B) a solution in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$
(C) no solution in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$
(D) a solution in $\left[\frac{3\pi}{4}, \pi\right)$

15. In which of the following interval(s) the value of the expression $y = 2(\sqrt{2} - 1)\sin x - 2\cos 2x + 2 - \sqrt{2}$ is negative?
(A) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ (B) $\left(0, \frac{\pi}{6}\right)$
(C) $\left(\frac{5\pi}{6}, \frac{5\pi}{4}\right)$ (D) $\left(\frac{7\pi}{4}, 2\pi\right)$

16. The equation $\sin\alpha = \tan(\alpha - \beta) + \cos\alpha\cdot\tan\beta$ holds true if
(A) $\alpha = n\pi + \beta$
(B) $\alpha = 2\nu\pi + 2\beta$
(C) $\alpha = 2n\pi$ and $\beta \in R$
(D) $\beta = 2\nu\pi$ and $\alpha \in R$
n is an integer

17. If $\cos 3\theta = \cos 3\alpha$ then the value of $\sin \theta$ can be given by

(A) $\pm \sin \alpha$ (B) $\sin\left(\frac{\pi}{3} \pm \alpha\right)$

(C) $\sin\left(\frac{2\pi}{3} + \alpha\right)$ (D) $\sin\left(\frac{2\pi}{3} - \alpha\right)$

18. Which of the following sets can be the subset of the general solution of the equation $1 + \cos 3x = 2 \cos 2x$?

(A) $n\pi + \frac{\pi}{3}$ (B) $n\pi + \frac{\pi}{6}$

(C) $n\pi - \frac{\pi}{6}$ (D) $2n\pi$

19. Which of the following equations have no solution?

(A) $2^{|x|} = \sin x^2$ (B) $3^{|\sin \sqrt{x}|} = |\cos x|$

(C) $x^2 = -\cos x$ (D) $3x^2 = 1 - 2\cos x$

20. Which of the following equations have their general solution equal to $2n\pi$ where $n \in I$?

(A) $\sin 3x + 3 \sin 4x + \sin 5x = 0$

(B) $2\tan^2 x + 3 = \frac{3}{\cos x}$

(C) $\cos x \cdot \cos 2x \cdot \cos 4x = 1$

(D) $\sin \frac{3x}{2} \cdot \cos \frac{x}{2} = \frac{1}{2} \sin 2x$

Comprehension-based Questions

Paragraph for question nos. 1 to 3

Equation of the form $P(\sin x \pm \cos x, \sin x \cos x) = 0$ where $P(y, z)$ is a polynomial, can be solved by the change: $\cos x \pm \sin x = t$; $1 \pm 2 \sin x \cos x = t^2$. Reduce the given equation into $P\left(t, \frac{t^2-1}{2}\right) = 0$.

1. $\sin x + \cos = 1 + \sin x \cos x$ is

(A) $\frac{\pi}{2} + 2\pi n$ and $2n\pi$

(B) $\frac{\pi}{4} + 2\pi n$ and $(2n + 1)\pi$

(C) $2n\pi$

(D) None of these

2. If $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$, then x is

(A) $n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{3}$

(C) $n\pi \pm \frac{\pi}{6}$ (D) $2n\pi \pm \frac{\pi}{6}$

3. $\sin^4 x + \cos^4 x = \sin x \cos x$ then x is

(A) $(6n + 1)\frac{\pi}{6}$ (B) $n\pi$

(C) $(4n + 1)\frac{\pi}{4}$ (D) None of these

Paragraph for question nos. 4 to 6

Consider an equation $\sin x + \sin y \le 2$ (1)

We know that $\sin x \le 1$ and $\sin y \le 1$ for all x, y.

So $\sin x + \sin y \le 2$ for all x and y.

Therefore, $\sin x + \sin y = 2$ if and only if $\sin x = 1$ and $\sin y = 1$

$\Rightarrow x = 2n\pi + \frac{\pi}{2}$ and $y = 2m\pi + \frac{\pi}{2}$.

Which is the required solution of given equation. To solve the equation (1), we have used the boundedness of $\sin x$ rather than using conventional methods of solving equations. In general we employ one or more of the following extreme value conditions.

(a) $-1 \le \sin x \le 1 \Rightarrow |\sin x| \le 1$ and $\sin^2 x \le 1$

(b) $-1 \le \cos x \le 1 \Rightarrow |\cos x| \le 1$ and $\cos^2 x \le 1$

(c) $-\sqrt{a^2 - b^2} \le a \sin x + b \cos x \le \sqrt{a^2 + b^2}$

$\Rightarrow |a \sin x + b \cos x| \le \sqrt{a^2 + b^2}$

4. The minimum value of $27^{\cos 2x}\, 81^{\sin 2x}$ is

(A) 1 (B) $\frac{1}{9}$

(C) $\frac{1}{81}$ (D) $\frac{1}{243}$

5. Number of roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $[0, 2\pi]$ is

(A) 0 (B) 1

(C) 2 (D) 4

6. The smallest positive number of p for which the equation $\cos(p \sin x) = \sin(p \cos x)$ has a solution in $[0, 2\pi]$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4\sqrt{2}}$ (D) $\frac{\pi}{2\sqrt{2}}$

Paragraph for question nos. 7 to 9

Let α, β, γ be positive integers in arithmetic progression such that $\alpha° < \beta° < \gamma° < 180°$.

Also $\sin \alpha° + \sin \beta° = \sin \gamma°$ and $\cos \alpha° - \cos \beta° = \cos \gamma°$.

7. The value of (tan α° + tan β° + tan γ°) is equal to
(A) 3 (B) $2-\sqrt{3}$
(C) 5 (D) $2\sqrt{2}+1$

8. Common difference of an arithmetic progression lies in the interval
(A) (2, 10) (B) (10, 18)
(C) (20, 25) (D) (25, 35)

9. Number of solutions of the equation cos x + cos $2x$ + cos $3x$ = tan α° + tan γ° – tan β° in the interval [0, 4π] is equal to
(A) 2 (B) 3
(C) 4 (D) 5

Paragraph for question nos. 10 and 11

Let $f(x) = \cos x + \sin x - 1$ and $g(x) = \sin 2x - 2$.

10. Number of solutions of the equation $f(x) = g(x)$ in $x \in \left[-\pi, \frac{7\pi}{2}\right]$ is equal to
(A) 4 (B) 5
(C) 6 (D) 8

11. If the equation $f(x) = k$ has atleast one real solution then the number of possible integral values of k is equal to
(A) 1 (B) 2
(C) 3 (D) 4

Matrix Match-type Questions

1. Match the following

Column - I	Column - II
(A) $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$	(P) $\sin x = \frac{1}{2}$
(B) $1 + \sin 2x = \cos x + \sin x$	(Q) $\tan x = -1$
(C) $4x^4 + x^6 + \sin^2 5x = 0$	(R) $x = 0$
(D) $\tan x = \frac{1}{\sqrt{3}}$	(S) $x = \frac{19\pi}{5}$

2. Match the following for number of solutions in $[0, 2\pi]$

Column - I	Column - II
(A) $\sin^2\theta - \tan^2\theta = 1$	(P) 2
(B) $\sin\theta + \cos\theta = 1$	(Q) 0
(C) $\tan\theta + \sec\theta = 2\cos\theta$	(R) 3
(D) $3\sin^2\theta - 4\sin\theta + 1 = 0$	(S) 1

3. Match the following

Column - I	Column - II
(A) If $x \in \left[0, \frac{\pi}{2}\right]$, the number of solutions of the equation, $\sin 7x + \sin 4x + \sin x = 0$ is	(P) 4
(B) Let $f(x) = \cos^2 x + \cos^2 2x + \cos^2 3x$. Number of values of $x \in [0, \pi]$ for which $f(x)$ equals the smallest positive integer is	(Q) 5
(C) Number of principal solution of the trigonometric equation $\tan x + \tan 2x + \tan x \cdot \tan 2x = 1$, is	(R) 6
(D) The number of points $P(x, y)$ lying inside or on the circle $x^2 + y^2 = 9$ and satisfying the equation $\tan^4 x + \cot^4 x + 2 = 4\sin^2 y$, is	(S) 7
(E) Let $0 < \theta_1 < \theta_2 < \theta_3 < \ldots\ldots$ denote the positive solution of the equation $3 + 3\cos\theta = 2\sin^2\theta$. If $\theta_3 + \theta_7 = a\pi$, where a is an integer then 'a' equals	(T) 8

4. Match the following

Column - I	Column - II
(A) If solution of the equation $x^{4+\log x} = 32$ (where base of logarithm is 2) are x_1 and x_2 ($x_1 > x_2$) then the value of $\frac{x_1}{x_2}$ is divisible by	(P) 2 (Q) 3
(B) Let $\cos(\alpha+\beta) = \frac{4}{5}$; $\sin(\alpha-\beta) = \frac{5}{13}$ and α, β lie between 0 and $\frac{\pi}{4}$. If the value of tan 2α is $\frac{56}{k}$, then k is divisible by	(R) 8
(C) The difference between maximum and minimum values of $\sin^2 x - 20\cos x + 1$ is divisible by	(S) 11

5. Match the following

Column - I	Column - II
(A) $\sin x \cdot \cos^3 x > \cos x \cdot \sin^3 x$, $0 \le x \le 2\pi$, is	(P) $\left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
(B) $4\sin^2 x - 8\sin x + 3 \le 0$, $0 \le x \le 2\pi$, is	(Q) $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$
(C) $\lvert \tan x \rvert \le 1$ and $x \in [-\pi, \pi]$ is	(R) $\left(0, \frac{\pi}{4}\right)$
(D) $\cos x - \sin x \ge 1$ and $0 \le x \le 2\pi$	(S) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

Integer-type Questions

1. For $0 < x, y < \pi$, find the number of ordered pairs (x, y) satisfying the system of equations $\cot^2(x-y) - (1+\sqrt{3})\cot(x-y) + \sqrt{3} = 0$ and $\cos y = \frac{\sqrt{3}}{2}$.

2. Find number of solutions of the equation $(\log_2 \cos\theta)^2 + \log_{\frac{4}{\cos\theta}}(16\cos\theta) = 2$, in interval $[0, 2\pi)$.

3. For $\alpha \in \left(\frac{3\pi}{4}, \pi\right)$ and $\beta \in R - (2n+1)\frac{\pi}{2}, n \in I$, find number of ordered pairs (α, β) satisfying the equation $\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}} = \frac{1+\cot\alpha}{\sqrt{\tan\beta}}$.

4. Find the number of principal solutions of the trigonometric equation $5\left(\sin x + \frac{\cos 3x + \sin 3x}{1 + 2\sin 2x}\right) = \cos 2x + 3$.

5. Find the number of points of intersection of the curves $y = \cos x$, $y = \sin 3x$, if $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.

6. Let N be the number of triplets (x, y, z) where $x, y, z \in [0, 2\pi]$ satisfying the inequality $(4 + \sin 4x)(2 + \cot^2 y)(1 + \sin^4 z) \le 12\sin^2 z$. Find the value of $\frac{N}{2}$.

7. Let $A = \{\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_m\}$, $B = \{\mu_1, \mu_2, \mu_3, \ldots\ldots\ldots, \mu_n\}$ are two sets of values of λ and μ where ($\lambda_i < \lambda_{i+1}$, $\mu_j < \mu_{j+1}$ $\forall$ i and j and $\lambda, \mu \in (0, 100\pi]$) such that for the ordered pair (λ_i, μ_j) the equation $x^2(\sin\lambda x + \cos\mu x) - 2(\sin(\lambda+\mu)x + \sin(\lambda-\mu)x)x + \sin\lambda x + \cos\mu x = 0$ has a positive solution. If $\sum_{i=1}^{m}\lambda_i + \sum_{j=1}^{n}\mu_i = \frac{5}{2}k\pi$ then find the value of k.

8. Consider an equation with x as a variable $7\sin 3x - 2\sin 9x = \sec^2\theta + 4\operatorname{cosec}2\theta$, then find the value of $\frac{15}{\pi}$((minimum positive root) – (maximum negative root)).

9. Find the sum of the first 100 rational terms of the increasing sequence of the positive roots of the equation $\sin(\pi x^2) - \sin(\pi(x^2 + 2x)) = 0$.

10. If the sum of all the angles which lie in $[0, 2\pi]$ and satisfying the equation $(8\cos 4\theta - 3)(\cot\theta + \tan\theta - 2)(\cot\theta + \tan\theta + 2) = 12$ is $k\pi$, find the value of k.

11. Find the number of solutions satisfying the equation $\cos\frac{x}{2} - \sin\frac{x}{2} = \sin x - 1$, where $\lvert x - \pi \rvert \le \frac{3\pi}{2}$.

12. Let U: denotes the value of $4\cos\frac{\pi}{5} - 4\cos\frac{2\pi}{5} + 4\sin\frac{\pi}{10}\cos\frac{\pi}{5}$ and V: denotes the number of values of $\theta \in [-2\pi, 2\pi]$ satisfying the equation $\operatorname{cosec}\theta = 1 + \cot\theta$. Find $(U + V)$.

13. Find the number of solutions of the equation $2\cos 3x(3 - 4\sin^2 x) = 1$ in $[0, 2\pi]$.

14. If the sum of all values of x satisfying the system of equations

$$\tan x + \tan y + \tan x \cdot \tan y = 5$$
$$\sin(x + y) = 4\cos x \cdot \cos y$$

is $\frac{k\pi}{2}$, where $x \in \left(0, \frac{\pi}{2}\right)$ then find the value of k.

15. Find the number of values of θ, $0 \le \theta \le 2\pi$ such that the graphs of $f(x) = \left(2\sin^2\frac{\theta}{2}\right)x^2 + \left(\cot\frac{\theta}{2}\right)x - 1$ and $g(x) = \left(2\cos^2\frac{\theta}{2}\right)x^2 - \left(\tan\frac{\theta}{2}\right)x + \cot^2\theta$ has exactly one point in common.

16. Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be all solutions of the equation $(\sin 3x)(\sin 3x - \cos x) = \sin x(\sin x - \cos 3x)$ where $0 \le x_i \le 2\pi\ \forall\ i$. If $\sum_{i=1}^{n} x_i = \frac{a\pi}{b}$ (where a and b are coprime) then find the value of $(a + b)$.

17. If the sum of all the principal solutions of the equation $2x - \tan\frac{\pi}{6}\cos 2x = 1$ is $\frac{k\pi}{12}$, then find the value of k.

18. Find the number of distinct solutions for the equation $\cos x \cos 2x \cos 3x \cos 4x \cos 5x = 0$ in the interval $[0, \pi]$.

19. Find the number of solutions of the equation $\sin 2\theta + \cos 2\theta + 4\sin\theta = 1 + 4\cos\theta$ lying in the interval $[-2\pi, 2\pi]$.

20. Sum of the principal solutions of the equation, $\sec x + \tan x = 2\cos x$ is ________.

21. Find the smallest positive number p for which the equation: $\cos(p\sin x) = \sin(p\cos x)$ has a solution $x \in [0, 2\pi]$.

22. Determine the smallest positive value of x which satisfy the equation, $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$.

JEE ARCHIVES

JEE Main

1. Find the number of roots of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$ **[AIEEE-2002]**

(A) 1 (B) 2
(C) 3 (D) 4

2. General solution of $\tan 5\theta = \cot 2\theta$ is **[AIEEE-2002]**

(A) $\theta = \frac{n\pi}{7} + \frac{\pi}{14}$

(B) $\theta = \frac{n\pi}{7} + \frac{\pi}{5}$

(C) $\theta = \frac{n\pi}{7} + \frac{\pi}{2}$

(D) $\theta = \frac{n\pi}{7} + \frac{\pi}{5}, n \in Z$

3. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is **[AIEEE-2006]**

(A) 6 (B) 1
(C) 2 (D) 4

4. If $0 < x < \pi$, and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is **[AIEEE-2006]**

(A) $\frac{(4-\sqrt{7})}{3}$ (B) $-\frac{(4+\sqrt{7})}{3}$

(C) $\frac{(1+\sqrt{7})}{4}$ (D) $\frac{(1-\sqrt{7})}{4}$

5. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has **[AIEEE-2012]**

(A) exactly one real root
(B) exactly four real root
(C) infinite number of real root
(D) no real root

6. In ΔPQR if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to **[AIEEE-2012]**

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{6}$ (D) $\frac{\pi}{6}$

7. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is **[JEE-2016]**

(A) 5 (B) 7
(C) 9 (D) 3

8. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 minutes from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, from B to reach the pillar, is **[JEE-2016]**

(A) 10 (B) 20
(C) 5 (D) 6

9. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is **[JEE-2017]**

(A) $-\frac{7}{9}$ (B) $-\frac{3}{5}$

(C) $\frac{1}{3}$ (D) $\frac{2}{9}$

10. If sum of all the solutions of the equation

$$8\cos x \cdot \left(\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$$

in $[0, \pi]$ is $k\pi$, then k is equal to **[JEE-2018]**

(A) $\frac{20}{9}$ (B) $\frac{2}{3}$

(C) $\frac{13}{9}$ (D) $\frac{8}{9}$

11. Let $f_k(x)=\frac{1}{k}(\sin^k x+\cos^k x)$ for $k = 1, 2, 3.....$ Then for all $x \in R$, the value of $f_4(x) - f_6(x)$ is equal to **[JEE-2019]**

(A) 1/12 (B) 1/4

(C) –1/12 (D) 5/12

12. If $\sin^4\alpha+4\cos^4\beta+2=4\sqrt{2}\sin\alpha\cos\beta;\alpha,\beta\in[0,\pi]$, then $\cos(\alpha+\beta)-\cos(\alpha-\beta)$ is equal to **[JEE-2019]**

(A) –1 (B) 0

(C) $-\sqrt{2}$ (D) $\sqrt{2}$

13. $\left(\dfrac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$ is **[JEE-2020]**

(A) $-\frac{1}{2}\left(1-i\sqrt{3}\right)$ (B) $-\frac{1}{2}\left(1-i\sqrt{3}\right)$

(C) $\frac{1}{2}\left(1-i\sqrt{3}\right)$ (D) $\frac{1}{2}\left(\sqrt{3}-i\right)$

14. If the equation $\cos^4\theta+\sin^4\theta+\lambda=0$ has real solution for θ, then λ lies in the interval **[JEE-2020]**

(A) $\left(-\frac{1}{2},-\frac{1}{4}\right)$ (B) $\left(-1,-\frac{1}{2}\right)$

(C) $\left(-\frac{3}{2},-\frac{5}{4}\right)$ (D) $\left(-\frac{3}{2},-1\right)$

15. If a ΔABC has vertices $A(-1, 7)$, $B(-7, 1)$ and $C(5, -5)$, then its orthocentre has coordinates **[JEE-2020]**

(A) (-3, 3) (B) $\left(-\frac{3}{5},\frac{3}{5}\right)$

(C) $\left(\frac{3}{5},-\frac{3}{5}\right)$ (D) (3, –3)

16. If $L=\sin^2\left(\frac{\pi}{16}\right)-\sin^2\left(\frac{\pi}{8}\right)$ and

$M=\cos^2\left(\frac{\pi}{16}\right)-\sin^2\left(\frac{\pi}{8}\right)$, then **[JEE-2020]**

(A) $M=\frac{1}{2\sqrt{2}}+\frac{1}{2}\cos\frac{\pi}{8}$

(B) $M=\frac{1}{4\sqrt{2}}+\frac{1}{4}\cos\frac{\pi}{8}$

(C) $L=-\frac{1}{2\sqrt{2}}+\frac{1}{2}\cos\frac{\pi}{8}$

(D) $L=\frac{1}{4\sqrt{2}}-\frac{1}{4}\cos\frac{\pi}{8}$

17. A ray of light coming from the point $(2, 2\sqrt{3})$ is incident at an angle 30° on the line $x = 1$ at the point A. The ray gets reflected on the line $x = 1$ and meets x-axis at the point B. Then, the line AB passes through the point **[JEE-2020]**

(A) $\left(4,-\sqrt{3}\right)$ (B) $\left(3,-\frac{1}{\sqrt{3}}\right)$

(C) $\left(3,-\sqrt{3}\right)$ (D) $\left(4,-\frac{\sqrt{3}}{2}\right)$

18. The angle of elevation of the top of a hill from a point on the horizontal plane passing through the foot of the hill is found to be 45°. After walking a distance of 80 meters towards the top, up a slope inclined at an angle of 30° to the horizontal plane, the angle of elevation of the top of the hill becomes 75°. Then the height of the hill (in meters) is _______. **[JEE-2020]**

19. The angle of elevation of the summit of a mountain from a point on the ground is 45°. After climbing up one km towards the summit at an inclination of 30° from the ground, the angle of elevation of the summit is found to be 60°. Then the height (in km) of the summit from the ground is **[JEE-2020]**

(A) $\frac{1}{\sqrt{3}+1}$ (B) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

(C) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (D) $\frac{1}{\sqrt{3}-1}$

20. If $0 < \theta, \phi < \frac{\pi}{2}$, $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \cdot \sin^{2n}\phi$ then **[JEE 2021]**

(A) $xy - z = (x + y)z$ (B) $xy + yz + zx = z$
(C) $xyz = 4$ (D) $xy + z = (x + y)z$

21. A man is observing, from the top of a tower, a boat speeding towards the tower from a certain point A, with uniform speed. At that point, angle of depression of the boat with the man's eye is 30° (Ignore man's height). After sailing for 20 seconds, towards the base of the tower (which is at the level of water), the boat has reached a point B, where the angle of depression is 45°. Then the time taken (in seconds) by the boat from B to reach the base of the tower is **[JEE 2021]**

(A) 10 (B) $10\sqrt{3}$
(C) $10(\sqrt{3}+1)$ (D) $10(\sqrt{3}-1)$

22. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, $n > 0$, then the value of n is equal to **[JEE 2021]**

(A) 20 (B) 12
(C) 9 (D) 16

23. If $15\sin^4\alpha + 10\cos^4\alpha = 6$, for some $\alpha \in$ R, then the value of $27\sec^6\alpha + 8\operatorname{cosec}^6\alpha$ is equal to **[JEE 2021]**

(A) 350 (B) 500
(C) 400 (D) 250

24. A spherical gas balloon of radius 16 meter subtends an angle 60° at the eye of the observer A while the angle of elevation of its center from the eyes of A is 75°. Then the height (in meter) of the top most point of the balloon from the level of the observer's eye is **[JEE 2021]**

(A) $8(\sqrt{2}+2+\sqrt{3})$ (B) $8(\sqrt{6}+\sqrt{2}+2)$
(C) $8(2+2\sqrt{3}+\sqrt{2})$ (D) $8(\sqrt{6}-\sqrt{2}+2)$

25. Two poles, AB of length a metres and CD of length $a + b$ $(b \neq a)$ metres are erected at the same horizontal level with bases at B and D. If BD = x and $\tan \angle ACB = \frac{1}{2}$, then **[JEE 2021]**

(A) $x^2 + 2(a + 2b)x - b(a + b) = 0$
(B) $x^2 + 2(a + 2b)x + a(a + b) = 0$
(C) $x^2 - 2ax + b(a + b) = 0$
(D) $x^2 - 2ax + a(a + b) = 0$

26. The number of solutions of the equation $|\cot x| = \cot x + \frac{1}{\sin x}$ in the interval $[0, 2\pi]$ is ________. **[JEE 2021]**

JEE Advanced

Single Correct Option-type Questions

1. The number of integral values of k for which the equation $7 \cos x + 5 \sin x = 2k + 1$ has a solution is **[JEE-2002]**

(A) 4 (B) 8
(C) 10 (D) 12

2. $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$, where $\alpha, \beta \in [-\pi, \pi]$, number of pairs of α, β which satisfy both the equations is **[JEE-2005]**

(A) 0 (B) 1
(C) 2 (D) 4

3. If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is **[JEE-2006]**

(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

(B) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$

(C) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

(D) $\left(\frac{41\pi}{48}, \pi\right)$

4. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is **[JEE-2007]**

(A) zero (B) one
(C) two (D) four

5. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then **[JEE-2011]**

(A) $P \subset Q$ and $Q - P \neq \varnothing$
(B) $Q \not\subset P$
(C) $P \not\subset Q$
(D) $P = Q$

6. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has **[JEE-2014]**

(A) infinitely many solutions
(B) three solutions
(C) one solution
(D) no solution

7. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to **[JEE-2016]**

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$
(C) 0 (D) $\frac{5\pi}{9}$

8. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to **[JEE-2016]**

(A) $3 - \sqrt{3}$ (B) $2(3 - \sqrt{3})$
(C) $2(\sqrt{3} - 1)$ (D) $2(2 + \sqrt{3})$

9. Let $\frac{\pi}{2} < x < \pi$ be such that $\cot x = \frac{-5}{\sqrt{11}}$. Then

$$\left(\sin\frac{11x}{2}\right)(\sin 6x - \cos 6x) + \left(\cos\frac{11x}{2}\right)(\sin 6x + \cos 6x)$$

is equal to: **[JEE Advanced 2024]**

(A) $\frac{\sqrt{11} - 1}{2\sqrt{3}}$ (B) $\frac{\sqrt{11} + 1}{2\sqrt{3}}$
(C) $\frac{\sqrt{11} + 1}{3\sqrt{2}}$ (D) $\frac{\sqrt{11} - 1}{3\sqrt{2}}$

Multiple Correct Option-type Questions

1. Let α and β be an nonzero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true? **[JEE-2017]**

(A) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(E) None of the above

2. Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If $\tan\frac{x}{2} + \tan\frac{z}{2} = \frac{2y}{x + y + z}$, then which of the following statements is/are TRUE? **[JEE-2020]**

(A) 2Y = X + Z (B) Y = X + Z
(C) $\tan\frac{x}{2} = \frac{x}{y + z}$ (D) $x^2 + z^2 - y^2 = xz$

Matrix Match-type Questions

1. Let $f(x) = \sin(\pi\cos x)$ and $g(x) = \cos(2\pi\sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order

$X = \{x : f(x) = 0\}$, $Y = \{x : f'(x) = 0\}$,
$Z = \{x : g(x) = 0\}$, $W = \{x : g'(x) = 0\}$,

Column-I contains the sets X, Y, Z and W. Column-II contains some information regarding these sets. **[JEE-2019]**

Column I	Column II
(I) X	(P) $\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II) Y	(Q) an arithmetic progression
(III) Z	(R) NOT an arithmetic progression
(IV) W	(S) $\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
	(T) $\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
	(U) $\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination?

(A) (III), (P), (Q), (U)
(B) (IV), (P), (R), (S)
(C) (III), (R), (U)
(D) (IV), (Q), (T)

2. Consider the following lists:

Column I	Column II
(I) $\left\{x \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right] : \cos x + \sin x = 1\right\}$	(P) has two elements
(II) $\left\{x \in \left[-\frac{5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$	(Q) has three elements
(III) $\left\{x \in \left[-\frac{6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos(2x) = \sqrt{3}\right\}$	(R) has four elements
(IV) $\left\{x \in \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$	(S) has five elements
	(T) has six elements

The correct option is: **[JEE Advanced 2022]**

(A) (I) → (P); (II) → (S); (III) → (P); (IV) → (S)
(B) (I) → (P); (II) → (P); (III) → (T); (IV) → (R)
(C) (I) → (Q); (II) → (P); (III) → (T); (IV) → (S)
(D) (I) → (Q); (II) → (S); (III) → (P); (IV) → (R)

Integer-type Questions

1. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm1, \pm2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is **[JEE-2010]**

2. The number of all possible values of θ when $\theta \in (0, \pi)$ for which the system of equation **[JEE-2010]**

$$(y + z)\cos 3\theta = (xyz)\sin 3\theta$$

$$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$

$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$ have a solution (x_0, y_0, z_0) with $y_0, z_0 \neq 0$ is

3. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3}a\cos x + 2b\sin x = c, x \,\varepsilon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then, the value of $\frac{b}{a}$ is

4. The number of distinct solution of equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is. **[JEE-2015]**

5. Let $f:[0,2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x))\sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2]: f(x) \geq 0\}$ $[\alpha, \beta]$, then the value of $\beta - \alpha$ is____ **[JEE-2020]**

ANSWER KEYS

Check Your Understanding-I

1. 4 **2.** 7

3. Min. value = 3^{-5} for $x = (4n-1)\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{3}{4}$, $n \in I$; Max. value = 3^5 for $x = (4n+1)\frac{\pi}{4} - \tan^{-1}\frac{3}{4}$, $n \in I$

4. ϕ **5.** 30°, 150°, 210°, 330°

Check Your Understanding-II

1. $x = n\pi + (-1)^n \frac{\pi}{6}$ and $y = m\pi \pm \frac{\pi}{6}$ where m and n are integers. **2.** $x = 2n\pi \pm \frac{2\pi}{3}, n \in I$

3. $\alpha - 2\pi$; $\alpha - \pi$, α, $\alpha + \pi$, where $\tan \alpha = \frac{2}{3}$ **4.** 30°, 45°, 90°, 135°, 150°

5. $x = 2n\pi$ *or* $x = nx + (-1)^n\left(-\frac{\pi}{2}\right)$ *or* $x = nx + (-1)^n \frac{\pi}{6}$ **6.** $\frac{(2n+1)\pi}{4}, k\pi$, where $n, k \in I$

Practice Exercises

1. **(a)** $n\pi + (-1)^n \frac{\pi}{4}, n \in I$ **(b)** $n\pi + \frac{\pi}{3} + 1, n \in I$ **(c)** $n\pi - \frac{\pi}{4}, n \in I$ **(d)** $n\pi + (-1)^n \frac{\pi}{3}, n \in I$ **(e)** $n\pi \pm \frac{\pi}{4}, n \in I$

2. $\frac{n\pi}{4}, n \in I$ or $\frac{(2n+1)\pi}{10}, n \in I$ **3.** $2n\pi \pm \frac{\pi}{3}, n \in I$

4. $\left(2n + \frac{1}{2}\right)\frac{\pi}{5}, n \in I$ or $2n\pi - \frac{\pi}{2}, n \in I$ **5.** $\left(n + \frac{1}{2}\right)\frac{\pi}{9}, n \in I$ **6.** $n\pi + \frac{\pi}{3}, n \in I$ or $n\pi + \frac{\pi}{4}, n \in I$

7. 45° and 60° **8.** $n\pi + (-1)^n \frac{\pi}{10}, n \in I$ or $n\pi - (-1)^n \frac{3\pi}{10}, n \in I$ **9.** $\left(n + \frac{1}{4}\right)\frac{\pi}{2}, n \in I$

10. $\frac{n\pi}{3}, n \in I$ or $\left(n \pm \frac{1}{3}\right)\pi, n \in I$ **11.** $2n\pi, n \in I$ or $\frac{2n\pi}{3} + \frac{\pi}{6}, n \in I$ **12.** 30°, 45°, 90°, 135°, 150°

13. $x = (2n+1)\frac{\pi}{4}, n \in I$ or $x\ (2n+1)\frac{\pi}{2}, n \in I$ or $x = n\pi \pm \frac{\pi}{6}, n \in I$

14. $m\pi, m \in I$ or $\frac{m\pi}{n-1}, m \in I$ or $\left(m + \frac{1}{2}\right)\frac{\pi}{n}, m \in I$ **15.** $n\pi + \frac{\pi}{6} + (-1)^n \frac{\pi}{4}, n \in I$

16. $2n\pi + \frac{2\pi}{3}, n \in I$ **17.** $2n\pi + \frac{\pi}{2}, n \in I$ or $2n\pi + 2\alpha$ where $\alpha = \tan^{-1}\frac{3}{7}, n \in I$

18. $n\pi \pm \frac{\pi}{6}, n \in I$ **19.** $\left(n + \frac{1}{3}\right)\frac{\pi}{3}, n \in I$ **20.** $x = \frac{n\pi}{3} - \frac{\pi}{9}, n \in I$ **21.** $\theta = n\pi + \frac{\pi}{4}$, $\phi = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

22. $\theta = \frac{7\pi}{12}, \frac{19\pi}{12}$ **23.** $x = 2n\pi + \frac{\pi}{6}, n \in I$ **24.** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$ and π

25. $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} - \frac{5\pi}{48}, n \in I$

26. $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$

27. A = 15º, B = 30º

28. $x = -1, y = n\pi \pm \frac{\pi}{4} + 1$

30. $n\pi; n\pi + (-1)^n \frac{\pi}{10}$ or $n\pi + (-1)^n \left(-\frac{3\pi}{10}\right)$

31. $n\pi + \frac{\pi}{8} < x < n\pi + \frac{\pi}{4}$

32. $\frac{1}{2}[n\pi + (-1)^n \sin^{-1}(1 - \sqrt{2a+3})]$ where $n \in I$ and $a \in \left[-\frac{3}{2}, \frac{1}{2}\right]$

33. $x = 2n\pi + \frac{3\pi}{4}, n \in I$

34. $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$

35. $x \in \left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right) \cup \left(2n\pi - \frac{\pi}{4}, 2n\pi\right) \cup \left((2n+1)\pi, 2n\pi + \frac{5\pi}{4}\right)$

36. $a = 0$ or $a < -1$

37. $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} + \frac{7\pi}{48}, n \in I$

38. $x = 2n\pi \pm \frac{2\pi}{3}, n \in I$

39. $\alpha - 2\pi; \alpha - \pi, \alpha, \alpha + \pi$, where $\tan \alpha = \frac{2}{3}$

40. $x = 2nx + \frac{\pi}{12}$ *or* $2n\pi + \frac{17\pi}{12}; n \in I$

Assessment

JEE Main

Level I

1. B	**2.** D	**3.** D	**4.** A	**5.** B	**6.** C	**7.** B	**8.** A	**9.** B	**10.** D
11. A	**12.** A	**13.** D	**14.** B	**15.** A	**16.** C	**17.** A	**18.** B	**19.** B	**20.** B
21. A	**22.** C	**23.** C	**24.** B	**25.** A	**26.** D	**27.** D	**28.** A	**29.** B	**30.** B
31. C	**32.** B	**33.** A	**34.** A	**35.** B					

Level II

1. B	**2.** C	**3.** B	**4.** C	**5.** A	**6.** B	**7.** D	**8.** D	**9.** C	**10.** B
11. C	**12.** B	**13.** C	**14.** B	**15.** D	**16.** A	**17.** C	**18.** B	**19.** A	**20.** A

Integer-type Questions

1. 0.52	**2.** 0.19	**3.** 4	**4.** 46	**5.** 3.5	**6.** 4	**7.** 564	**8.** 13	**9.** 8

JEE Advanced

Single Correct Option-type Questions

1. D	**2.** C	**3.** C	**4.** D	**5.** C	**6.** D	**7.** B	**8.** B	**9.** D	**10.** B
11. C	**12.** A	**13.** D	**14.** A	**15.** B	**16.** D	**17.** B	**18.** B	**19.** B	**20.** C
21. B	**22.** A	**23.** C	**24.** A	**25.** C	**26.** C	**27.** B	**28.** B		

Multiple Correct Option-type Questions

1. A, B, C, D	**2.** A, D	**3.** B, D	**4.** A, B, C	**5.** A, C	**6.** B, C, D	**7.** A, B, C, D
8. A, B, C, D	**9.** A, C	**10.** C, D	**11.** B, C	**12.** B, C	**13.** A, D	**14.** A, B, C, D
15. B, C, D	**16.** A, B, C	**17.** A, C, D		**18.** B, C, D	**19.** A,C, D	**20.** B, C

Comprehension-based Questions

1. A 2. B 3. C 4. D 5. D 6. D 7. C 8. D 9. B 10. C
11. C

Matrix Match-type Questions

1. (A)–(P), (B)–(Q), (C)–(R), (D)–(S)
2. (A)–(Q), (B)–(R), (C)–(P), (D)–(R)
3. (A) P; (B) Q; (C) P; (D) T; (E) R
4. (A) P, R; (B) Q, S; (C) P, R
5. (A) R; (B) S; (C) P; (D) Q

Integer-type Questions

1. 2 2. 3 3. 0 4. 2 5. 3 6. 8 7. 2010 8. 10 9. 5050 10. 8
11. 4 12. 5 13. 10 14. 1 15. 4 16. 19 17. 40 18. 13 19. 4
20. 3.14 21. 1.11 22. 0.19

JEE Archives

JEE Main

1. B 2. A 3. D 4. B 5. D 6. C 7. B 8. C 9. A 10. C
11. A 12. C 13. C 14. B 15. A 16. A 17. C 18. 80 19. D 20. D
21. C 22. B 23. D 24. B 25. C 26. 1

JEE Advanced

Single Correct Option-type Questions

1. B 2. D 3. A 4. C 5. D 6. D 7. C 8. C 9. B

Multiple Correct Option-type Questions

1. E 2. B, C

Matrix Match-type Questions

1. B 2. B

Integer-type Questions

1. 3 2. 3 3. 0.5 4. 8 5. 1